Commutative and noncommutarive stochastic calculus : theory and applications

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Chapitre 1

Introduction

Ce manuscrit de thèse se compose de quatre chapitres. Le premier chapitre est réservé à une introduction générale dans la quelle nous présentons les principales motivations, les objectifs de notre étude et les résultats significatifs obtenus dans ce travail.

Le deuxième chapitre est consacré au calcul stochastique variationnel en temps discret sur la classe des martingales normales à d dimensions satisfaisant la propriété de représentation chaotique (PRC).

L'objet du troisième chapitre est la description de la mesure spectrale du processus de Jocobi libre.

Ces deux chapitres ont pour point commun l'utilisation du calcul stochastique (commutatif et non-commutatif) et son application au calcul des probabilités.

Le dernier chapitre qui est plutôt de nature analytique, contient une nouvelle approche permettant de décrire la mesure spectrale du mouvement Brownien (noté MB) sur $Gl(d, \mathbb{C})$.

1.1 Motivation

En temps continu, il est bien connu que le MB et le processus de Poisson sont les seuls processus à accroissemnts stationnaires et indépendants ayant la PRC (voir par exemple [32]). Un autre exemple de martingales continues ayant la PRC est celui des martingales d'Azéma, comme l'a démontré Émery en 1988 ([16]). P. A. Meyer a étudié la PRC sur la classe des martingales normales à d dimensions. Il a remarqué qu'elles donnent lieu à des intégrales itérées qui s'identifient aux éléments de l'espace de Fock, fournissant une injection isométrique canonique de l'espace de Fock dans $L^2(\Omega)$ ([30]). Une condition nécessaire pour que cette injection soit un isomorphisme (i.e. que ces martingales satisfont la PRC) est qu'elles vérifient une équation de structure (ES). D'ailleurs cette dernière condition est nécessaire pour la propriété, plus faible, de représentation prévisible (PRP).

En temps discret, les trois propriétés de représentation chaotique, de représentation prévisible et d'existence d'équation de structure sont équivalentes ([2]). Ceci nous amène à l'étude d'une classe importante de martingales normales, dites marches aléatoires obtuses pour lesquelles la suite des accroissements admet $d \geq 1$ valeurs possibles. Dans le cas unidimensionnel $d = 1$, ces martingales sont des marches de Bernoulli, ce qui permet donc de retrouver les résultats sur l'analyse stochastique de l'espace de Bernoulli (voir l'article [35] de N. Privault).

Une motivation du deuxième chapitre est d'avoir une meilleure intuition pour le cas beaucoup plus difficile des martingales à temps continu, comme dans le papier d'Émery [17]. Une autre raison de notre étude est l'espoir d'appliquer nos outils d'analyse stochastique multidimensionelle au cas non-commutatif, citons par exemple le modèle d'interactions quantiques répétées ([1]).

Dans le troisième chapitre, nous nous intéréssons, comme nous l'avons indiqué, à la description de la mesure spectrale du processus de Jacobi libre. Ce processus introduit par N. Demni dans [12] peut être considéré comme une extension du processus de Jacobi réel en dimension infinie. En effet, ce dernier peut s'interpréter, pour des valeurs entières de ses paramètres, comme étant la partie radiale d'une projection du mouvement Brownien sur la sphère unité. A partir de cette interprétation géométrique, Y. Doumerc a introduit le processus de Jacobi matriciel comme la partie radiale d'un coin supérieur gauche du MB sur le groupe unitaire ([14]). Celui-ci converge au sens des moments non commutatifs, vers ce qu'on appelle le MB unitaire libre quand la taille de la matrice tend vers l'infini ([5, 36]). En vertu de la liberté asymptotique du MB unitaire et des deux projections orthogonales ([20]), le processus de Jacobi libre peut être réalisé comme limite au sens des moments non commutatifs du processus de Jacobi matriciel. On peut aussi définir le processus de Jacobi libre de façon abstraite : considérons un MB unitaire libre Y dans une algèbre de von Neumann $\mathcal{A}$ et P, Q deux projections orthogonales de $\mathcal{A}$ libres avec Y, de rangs respectifs $\lambda\theta$ et θ et telles que $PQ = QP = P$ si $\lambda \leq 1$ et $PQ = QP = Q$ sinon. Le prossesus de Jacobi libre est défini comme $J \triangleq PYQY^*P$. Ainsi, ce processus est une famille d'opérateurs positifs bornés dans l'algèbre compressée $P\mathcal{A}P$. Par conséquent, sa mesure spectrale est entièrement déterminée par la suite de ses moments. Par ailleurs, on peut voir que la loi de J est celle du produit de deux projections ortogonales. Ceci est d'une importance capitale car dans ce cas, J est considéré comme un cas particulier du processus de libération associé à deux projections (dans notre situation les projections commutent) défini par D.V. Voiculescu dans le but de résoudre le problème d'isomorphisme des facteurs des groupes libres.

Dans le quatrième chapitre, on retrouve la description de la mesure spectrale qui apparaît lorsqu'on considère la partie radiale du MB sur $Gl(d, \mathbb{C})$, à la limite normalisée quand $d \to \infty$. Cette mesure est l'analogue, en probabilités libres, de la loi log-normale (voir [21, p. 5]) et a été décrite dans [7]. En particulier son support est compact et ses moments sont donnés par :

$$L_{n-1}^{(1)}(nt)\frac{e^{-nt/2}}{n} = \lim_{d\to\infty} \frac{1}{d}\mathbb{E}(\mathrm{tr}[(Z_{-t/d}^{\star} Z_{-t/d})^n)]), \quad n \geq 1, t > 0$$

où $L_n^{(1)}$ est le n-ième polynôme de Laguerre et Z est un MB sur $Gl(d, \mathbb{C})$. Notre approche est une adaptation de celle utilisée dans [13] afin de retrouver la description de la mesure spectrale du MB unitaire libre.

1.2 Présentation des résultats

1.2.1 Marches aléatoires obtuses

Dans l'étude de l'analyse stochastique des marches aléatoires obtuses, nous généralisons les résultats du calcul stochastique sur les fonctionelles de Bernoulli. Notre travail peut être divisé principalement en deux parties :

- La première partie est consacrée au calcul stochastique variationnel sur les fonctionnelles des marches aléatoires obtuses. On rappelle qu'une martingale d dimensionnelle $Z = (Z^1, \ldots, Z^d)$ est dite normale si

  $$\mathbb{E}[\Delta Z_n^i \Delta Z_n^j \,|\mathcal{F}_{n-1}] = \delta^{ij}, \quad i, j \in \{1, ..., d\},$$

  où $(\mathcal{F}_n)_{n\geq 0}$ est la filtration naturelle associée à Z. Dans un premier temps, on donne une construction de l'intégrale stochastique discrète par rapport à une martingale normale d dimensionnelle, ainsi que l'intégrale stochastique multiple associée

  $$\mathcal{I}^r(f_r) = \sum_{k_1,\ldots,k_r=1}^{d} \sum_{(i_1,\ldots,i_r)\in\Delta_r} f_r^{k_1,\ldots,k_r}(i_1, ..., i_r)\Delta Z_{i_1}^{k_1} ... \Delta Z_{i_r}^{k_r}$$

  avec $\Delta_r = \{(i_1, ..., i_r) \in \mathbb{N}^r, i_l \neq i_k, 1 \leq l < k \leq r\}$ et $f_r \in L^2(\Delta_r, \mathbb{R}^{d^r})$ une fonction symétrique. Puis, on s'intéresse à l'étude des martingales normales satisfaisant la PRC. De manière équivalente, ces martingales vérifient une équation de structure i.e.

  $$[\Delta Z_n^i, \Delta Z_n^j] = \delta^{ij} + \sum_k \Phi_k^{ij}(n)\Delta Z_n^k,$$

  où les Φ_k^{ij} sont des d^3 processus prévisibles. Celles-ci donnent lieu à des marches aléatoires obtuses ([2]).

Nous présentons une construction de l'opérateur gradient D sur les chaos de Wiener correspondant. Puis ce dernier est réalisé comme opérateur de différence finie. Il est utilisé, ensuite, pour démontrer une formule de représentation prévisible pour les variables aléatoires du type Clark-Ocone. Enfin, nous définissons l'opérateur divergence, adjoint de D, et nous montrons qu'il coïncide avec l'intégrale stochastique sur les processus prévisibles de carré sommables.

- La deuxième partie est consacrée à quelques applications des outils développés dans la première. On utilise le semi-groupe de Ornstein-Uhlenbeck, pour prouver deux identités de covariance. Ensuite, à l'aide de l'interprétation probabiliste de l'opérateur gradient on donne une représentation intégrale du semi-groupe d'Ornstein-Uhlenbeck à l'aide du noyau

  $$q_t^N(w,\widetilde{w}) = \prod_{i=0}^{N} \left(1 + e^{-t} < \Delta Z_i(w), \Delta Z_i(\widetilde{w}) >\right), \quad N \geq 1.$$

  Celle-ci est utilisée avec la formule de représentation de covariance pour prouver une inégalité de déviation pour les variables aléatoires bornées. De plus, motivé par les travaux de Dalang, Morton et Willinger ([10]) qui ont découvert en temps discret l'équivalence entre non-arbitrage et existence d'une mesure-martingale, on a eu recours à la formule de Clark-Ocone dans le but d'obtenir une solution au problème de couverture des options en marché complet. On donne en particulier une expression explicite d'une stratégie de couverture pour des modèles financiers dont le processus de prix est une marche obtuse.

1.2.2 Processus de Jacobi libre

En ce qui concerne le processus de Jacobi libre, on établit dans un premier temps une équation récurrente reliant la suite des moments. D'une manière équivalente, on écrit une EDP non linéaire pour la fonction génératrice des moments. Dans un second temps, on s'intérresse à la description de la mesure spectrale du processus de Jacobi libre μ_t, quand le rang des projecteurs vaut $1/2$. Celle-ci coïncide avec celle du cosinus du MB unitaire libre Y

$$\frac{1}{4}[Y_{2t}^{-1} + 2 + Y_{2t}]$$

modulo le changement de temps $t \to 2t$. On rappelle à ce propos que la description de la mesure de Y apparaît dans [7, 13]. Ainsi, μ_t est absolument continue par rapport à la mesure de Lebesgue et elle est donnée par

$$\mu_t(dx) = 2\frac{k_{2t}(e^{2i\arccos(\sqrt{x})})}{\sqrt{x(1-x)}}\mathbf{1}_{[0,1]}(x)dx,$$

où k_t est celle de Y. Deux preuves de ce résultat sont écrites : la première repose sur la résolution explicite de l'EDP alors que la deuxième est de nature combinatoire. Pour des rangs arbitraires des projecteurs, cette dernière a le mérite de donner une description presque complète de la mesure spectrale quand le deuxième paramètre varie tandis que le premier reste fixe. On achève cette partie par la donnée d'une décomposition de la fonction génératrice des moments en la somme de trois fonctions analytiques sur un voisinage de l'origine, dans le cas où le premier paramètre varie et le deuxième reste fixe.

1.2.3 MB sur $Gl(d, \mathbb{C})$

Dans le quatrième chapitre, on a revisité la description de la mesure spectrale ν_t qui apparaît à la limite quand $d \to \infty$, lorsqu'on considère les moments de la partie radiale du MB sur $Gl(d, \mathbb{C})$. On part dans un premier temps d'une représentation intégrale de la suite des moments sur une courbe de Jordan γ autour de l'origine

$$u_n(t) = -\frac{1}{2i\pi} \int_\gamma [g_t(z)]^n z \frac{\partial_z g_t(z)}{g_t(z)} dz, \quad g_t(z) := e^{-t(z+(1/2))} \left(1 + \frac{1}{z}\right).$$

Puis on montre (pour tout t négatif) l'existence d'une unique courbe de Jordan γ_t autour de l'origine telle que $g_t(\gamma_t) \in \mathbb{R}_+$. Cette courbe coïncide, modulo quelques transformations élémentaires, avec la frontière de la région Ω_t décrite dans [7, p. 271]. Ainsi, ν_t est la mesure image de

$$-z \frac{\partial_z g_t(z)}{g_t(z)} 1_{\gamma_t}(z) \frac{dz}{2i\pi}.$$

Un calcul direct permet donc de retrouver la desciption de ν_t qui figure dans [7].

Chapitre 2

Stochastic Analysis for obtuse random walks

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Abstract : We present a construction of the basic operators of stochastic analysis (gradient and divergence) for a class of discrete-time normal martingales called obtuse random walks. The approach is based on the chaos representation property and discrete multiple stochastic integrals. We show that these operators satisfy similar identities as in the case of the Bernoulli randoms walks. We prove a Clark-Ocone-type predictable representation formula, obtain two covariance identities and derive a deviation inequality. We close the exposition by an application to option hedging in discrete time.

2.1 Introduction

A celebrated Theorem of Wiener [41] (who introduced the terms 'homogeneous chaos' and 'polynomial chaos' in that paper) asserts that the chaotic representation property (hereafter CRP) holds for the Brownian motion. This property says that any square integrable random variable measurable with respect to a Brownian motion X can be expressed as an orthogonal sum of multiple stochastic integrals with respect to X. The main feature of this property is that it gives rise to an isometry between the Fock space and the L^2-space associated with this Brownian motion. In particular, the n-th Wiener chaos is identified to an element of the Fock space for any $n \geq 1$, opening the way to set up anticipating and non commutative stochastic calculus. Actually, Wiener chaoses were slightly different from the modern ones, introduced later by Itô [24] and studied by Meyer [31] for an interesting class of martingales called normal martingales (including Brownian motion and the compensated Poisson process). In addition to the martingale property, these processes, say $X = (X^1, \dots, X^d)$, are specified by the requirement

$$< X^i, X^j >_t = \delta^{ij} t.$$

Besides, Meyer noticed that any normal martingale gives rise to an isometry between the Fock space and its L^2-space. When this isometry is an isomorphism of Hilbert spaces, then it leads to structure equations (hereafter SE) i.e. we have

$$[X^i, X^j] = \delta^{ij} t + \int_0^t \sum_k (\Phi_k^{ij})_s dX_s$$

for some predictable processes (Φ_k^{ij}), and furthermore, X enjoys the PRP. However, this is far from being a sufficient condition unlike its analog in the discrete-time setting introduced and developed by Attal and Emery in [2]. In fact, in the discrete time case, Attal and Emery showed that SEs are necessary and sufficient for the CRP to hold (and also for the predictable representation property (PRP)). We are thus led to the so-called obtuse random walks which are a class of d-dimensional normal martingales such that the sequence of their increments $(\Delta X_n)_{n \geq 0}$ take $d+1$ values for each n. This fact translates that the filtration $(\mathcal{F}_n)_{n \geq 0}$ generated by X is of multiplicity $d+1$ (i.e. we move from $\mathcal{F}_n$ to $\mathcal{F}_{n+1}$ by decomposing of each atom of $\mathcal{F}_n$ to $d+1$ atoms) c.f. [2]. For $d = 1$, these random walks reduce to the Bernoulli process which was used in [35] to deal with the CRP and to define discrete multiple stochastic integrals with respect to a discrete-time normal martingale with i.i.d. sequences of increments.

In this paper, we shall focus on discrete time normal d-dimensional martingales (see definition below). Our main concern is generalizing the stochastic analysis for Bernoulli random walks (see [35]) to the obtuse random walks ($d > 1$) with not necessarily independent increments. First, we present a construction of the stochastic integral of predictable square-integrable processes and the associated multiple stochastic integrals of symmetric functions on $\mathbb{N}^n$ ($n \geq 1$), with respect to such martingales. Indeed, these iterated stochastic integrals give an isometry between this L^2-space and the Fock space. Next, we present a construction of the basic operators of stochastic analysis (gradient and divergence, [35]). We give a probabilistic interpretation of the gradient operator and prove that the divergence operator coincides with the stochastic integral on square summable predictable processes. These operators are used to derive a Clark-Ocone-type predictable representation formula and also to prove a deviation inequality. Finally, we apply the tools developed in this paper to discrete market models in order to obtain explicit expressions for hedging strategies [26, 35].

One motivation for this paper is of course to get a better intuition for the much more difficult continuous-time case, like in Émery's paper [17]. But, so far we have no new results in this direction. Another reason for our study is the hope to apply our tools of multi-dimensional discrete-time stochastic analysis to non-commutative discrete-time stochastic calculus, e.g. in models of repeated quantum interaction [1].

This paper is organized as follows. In section 2 we present a construction of the stochastic integral of predictable square-integrable processes with respect to a normal martingale. In the next section we construct the associated multiple stochastic integrals of functions f_n that are symmetric in n variables. In section 4 we present a characterization of obtuse random walks in discrete-time setting. The proof of the CRP is reviewed in section 5. A gradient operator D acting by finite differences is introduced in section 6 in connection with multiple stochastic integrals and is used in section 7 to state a Clark-Ocone-type predictable representation formula. The divergence operator δ is defined in section 8 as an extension of the discrete-time stochastic integral and we shall prove that it is the adjoint of D. The Ornstein-Uhlenbeck semi-group is used in section 9 to express a covariance identity. In section 10 we prove a deviation inequality for functionals of obtuse random walks. The last section is devoted to present a complete market model in discrete time as an application of the Clark-Ocone Formula.

2.2 Discrete stochastic integrals

Consider a discrete d-dimensional process $Y = (Y^1, ..., Y^d)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $(\mathcal{F}_n)_{n\geq 0}$ denote the filtration generated by $(Y_n)_{n\in\mathbb{N}}$ and $\mathcal{F}_{-1} = \{\varnothing, \Omega\}$. Recall that a d-dimensional integrable process is said to be an $\mathcal{F}_n$-martingale if each coordinate is so.

We recall also (see for example [2]) that a d-dimensional martingale $(Y_0+\ldots+Y_n)_{n\in\mathbb{N}}$ is said to be normal martingale if for any $n \geq 0$,

$$\mathbb{E}[Y_n^i Y_n^j \,|\mathcal{F}_{n-1}] = \delta^{ij}, \quad i, j \in \{1, ..., d\},$$

which can be written as

$$\mathbb{E}[Y_n \otimes Y_n \,|\mathcal{F}_{n-1}] = I_n,$$

where $\otimes$ is the Kronecker tensor product of the vector Y_n by itself.

In the sequel, we denote $< x, y >$ the inner product of x and y in $\mathbb{R}^d$, and we make following assumptions on $Y = (Y^1, ..., Y^d)$:

$$\mathbb{E}[Y_n^i \,|\mathcal{F}_{n-1}] = 0 \quad and \quad \mathbb{E}[Y_n^i Y_n^j \,|\mathcal{F}_{n-1}] = \delta^{ij}.$$

These assumptions imply that the process $(Y_0 + ... + Y_n)_{n\geq 0}$ is a normal martingale in the discrete time.

Definition 2.2.1. *Let $U = (U^1, ..., U^d)$ be a uniformly bounded sequence of random variables with finite support in $\mathbb{N}$ (i.e. $U_n = 0_{\mathbb{R}^d}$ except perhaps a finite number of indices). The stochastic integral of U with respect to Z is defined as*

$$\mathcal{I}(U) = \sum_{k=1}^{d} \sum_{n=0}^{\infty} U_n^k Y_n^k = \sum_{n=0}^{\infty} < U_n, Y_n > .$$

Now we recall that :

Definition 2.2.2. *A stochastic process $(X_n)_n$ is said to be predictable process with respect to $(\mathcal{F}_n)_n$ if X_n is $\mathcal{F}_{n-1}$-measurable for each n.*

Then one has the following result.

Proposition 2.2.3. *The stochastic integral extends to square-integrable predictable processes via the (conditional) isometry formula :*

$$\forall n \in \mathbb{N}, \quad \mathbb{E}[|\mathcal{I}(1_{[n,\infty)}U)|^2 \,|\mathcal{F}_{n-1}] = \mathbb{E}[\|1_{[n,\infty)}U\|^2 \,|\mathcal{F}_{n-1}]$$

where $1_{[n,\infty)}U$ denotes the process $(0, ..., 0, U_n, U_{n+1}, ...)$.

Proof. Let U, V be bounded predictable processes with finite support in $\mathbb{N}$, we have

$$\begin{aligned}
\mathbb{E}\left[\sum_{k=n}^{\infty} < U_k, Y_k > \sum_{l=n}^{\infty} < V_l, Y_l > |\mathcal{F}_{n-1}\right] &= \mathbb{E}\left[\sum_{i,j=1}^{d} \sum_{k,l=n}^{\infty} U_k^i Y_k^i V_l^j Y_l^j \,|\mathcal{F}_{n-1}\right] \\
&= \sum_{i,j=1}^{d} \sum_{k=n}^{\infty} \mathbb{E}\left[\mathbb{E}[U_k^i Y_k^i V_k^j Y_k^j \,|\mathcal{F}_{k-1}] \,|\mathcal{F}_{n-1}\right] \\
&\quad + \sum_{i,j=1}^{d} \sum_{n \le k < l} \mathbb{E}\left[\mathbb{E}[U_k^i Y_k^i V_l^j Y_l^j \,|\mathcal{F}_{l-1}] \,|\mathcal{F}_{n-1}\right] \\
&\quad + \sum_{i,j=1}^{d} \sum_{n \le l < k} \mathbb{E}\left[\mathbb{E}[U_k^i Y_k^i V_l^j Y_l^j \,|\mathcal{F}_{k-1}] \,|\mathcal{F}_{n-1}\right] \\
&= \sum_{i,j=1}^{d} \sum_{k=n}^{\infty} \mathbb{E}\left[U_k^i V_k^j \mathbb{E}[Y_k^i Y_k^j \,|\mathcal{F}_{k-1}] \,|\mathcal{F}_{n-1}\right] \\
&\quad +2 \sum_{i,j=1}^{d} \sum_{n \le l < k} \mathbb{E}\left[U_k^i Y_k^i V_l^j \mathbb{E}[Y_l^j \,|\mathcal{F}_{k-1}] \,|\mathcal{F}_{n-1}\right] \\
&= \sum_{i=1}^{d} \sum_{k=n}^{\infty} \mathbb{E}\left[U_k^i V_k^j \,|\mathcal{F}_{n-1}\right] \\
&= \mathbb{E}\left[\sum_{k=n}^{\infty} < U_k, V_k > |\mathcal{F}_{n-1}\right].
\end{aligned}$$

$\square$

2.3 Discrete multiple stochastic integrals

This section is devoted to the construction and to state the main properties of the multiple stochastic integrals of symmetric functions on $\mathbb{N}^r, r \geq 1$. We denote

$$\Delta_r = \{(i_1, ..., i_r) \in \mathbb{N}^r, i_l \neq i_k, 1 \leq l < k \leq r\}$$

and

$$f_r : \begin{array}{ccc} \Delta_r & \longrightarrow & \mathbb{R}^{d^r} \\ (i_1, ..., i_r) & \longmapsto & \left(f_r^{k_1,...,k_r}(i_1, ..., i_r)\right)_{1\leq k_1,...,k_r\leq d} \end{array}$$

a symmetric function in r variables.

Given $f_1 \in l^2(\mathbb{N})$ we let

$$\mathcal{I}^1(f_1) = \mathcal{I}(f_1) = \sum_{n=0}^{\infty} < f_1(n), Y_n > .$$

Definition 2.3.1. *For $r \geq 1$, the multiple stochastic integral of $f_r \in L^2(\Delta_r, \mathbb{R}^{d^r})$ with respect to the normal martingale $(Y_0 + \ldots + Y_n)_{n\geq 0}$ is defined by*

$$\mathcal{I}^r(f_r) = \sum_{k_1,...,k_r=1}^{d} \sum_{(i_1,...,i_r)\in\Delta_r} f_r^{k_1,...,k_r}(i_1, ..., i_r) Y_{i_1}^{k_1} ... Y_{i_r}^{k_r}.$$

Remark 2.3.2. *We take $\Delta_0 = \{0\}$, $L^2(\Delta_0, \mathbb{R}) = \mathbb{R}$ and define $\mathcal{I}^0(f_0) \equiv f_0$, $f_0 \in \mathbb{R}$.*

The following result gives a recurrence relation for multiple stochastic integral.

Proposition 2.3.3. *Let $r \geq 1$, we have*

$$\mathcal{I}^r(f_r) = r \sum_{k_r=1}^{d} \sum_{i_r=0}^{\infty} \mathcal{I}^{r-1}\left(f_r^{k_r}(*, i_r) 1_{[\![0,i_r-1]\!]^{r-1}}(*)\right) Y_{i_r}^{k_r},$$

where

$$f_r^k(*, i) : \begin{array}{ccc} \Delta_{r-1} & \longrightarrow & \mathbb{R}^{d^{r-1}} \\ (i_1, ..., i_{r-1}) & \longmapsto & \left(f_r^{k_1,...,k_{r-1},k}(i_1, ..., i_{r-1}, i)\right)_{1\leq k_1,...,k_{r-1}\leq d} \end{array}$$

Proof. We write

$$\begin{aligned}
\mathcal{I}^r(f_r) &= r! \sum_{k_1,...,k_r=1}^{d} \sum_{i_r=0}^{\infty} \sum_{0\le i_{r-1}\le i_r} ... \sum_{0\le i_1\le i_2} f_r^{k_1,...,k_r}(i_1,...,i_r) Y_{i_1}^{k_1}...Y_{i_r}^{k_r} \\
&= r \sum_{k_r=1}^{d} \sum_{i_r=0}^{\infty} \Bigg((r-1)! \sum_{k_1,...,k_{r-1}=1}^{d} \sum_{0\le i_{r-1}\le i_r} ... \\
&\quad ... \sum_{0\le i_1\le i_2} f_r^{k_1,...,k_r}(i_1,...,i_r) Y_{i_1}^{k_1}...Y_{i_{r-1}}^{k_{r-1}} \Bigg) Y_{i_r}^{k_r}.
\end{aligned}$$

□

The next proposition states an isometry formula

Proposition 2.3.4. *Let $r, s \ge 1$, and consider*

$$f_r = (f_r^{k_1,...,k_r}(i_1,...,i_r))_{1\le k_1,...,k_r\le d} \in L^2(\Delta_r, \mathbb{R}^{d^r}),$$

$$g_s = (g_s^{t_1,...,t_s}(j_1,...,j_s))_{1\le t_1,...,t_s\le d} \in L^2(\Delta_s, \mathbb{R}^{d^s}).$$

We have

$$\mathbb{E}\left[\mathcal{I}^r(f_r)\mathcal{I}^s(g_s)\right] = 1_{\{s=r\}} r! \sum_{k_1,...,k_r=1}^{d} < f_r^{k_1,...,k_s}, g_s^{k_1,...,k_s} >$$

Proof.

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{I}^r(f_r)\mathcal{I}^s(g_s)\right] \\
&= \sum_{\substack{k_1,...,k_r=1 \\ t_1,...,t_s=1}}^{d} \sum_{\substack{(i_1,...,i_r)\in\Delta_r \\ (j_1,...,j_s)\in\Delta_s}} f_r^{k_1,...,k_r}(i_1,...,i_r) g_s^{t_1,...,t_s}(j_1,...,j_s) \mathbb{E}[Y_{i_1}^{k_1}...Y_{i_r}^{k_r} Y_{j_1}^{t_1}...Y_{j_s}^{t_s}] \\
&= (r!)^2 \sum_{\substack{k_1,...,k_r=1 \\ t_1,...,t_s=1}}^{d} \sum_{\substack{0\le i_1<...<i_r \\ 0\le j_1<...<j_s}} f_r^{k_1,...,k_r}(i_1,...,i_r) g_s^{t_1,...,t_s}(j_1,...,j_s) \mathbb{E}[Y_{i_1}^{k_1}...Y_{i_r}^{k_r} Y_{j_1}^{t_1}...Y_{j_s}^{t_s}]
\end{aligned}$$

Note that if $r = s$ and $0 \le i_1 < ... < i_r$ and $0 \le j_1 < ... < j_r$ we have

$$\mathbb{E}[Y_{i_1}^{k_1}...Y_{i_r}^{k_r} Y_{j_1}^{t_1}...Y_{j_s}^{t_s}] = 1_{\{i_1=j_1,...i_r=j_r\}} 1_{\{k_1=t_1,...k_r=t_r\}},$$

hence we get

$$\mathbb{E}\left[\mathcal{I}^r(f_r)\mathcal{I}^r(g_r)\right]$$
$$= r! \sum_{\substack{k_1,\dots,k_r=1 \\ t_1,\dots,t_s=1}}^{d} \sum_{(i_1,\dots,i_r)\in\Delta_r} f_r^{k_1,\dots,k_r}(i_1,\dots,i_r) g_r^{k_1,\dots,k_r}(i_1,\dots,i_r) 1_{\{k_1=t_1,\dots k_r=t_r\}}.$$

If $r < s$ then there necessarily exists $k \in \{1, ..., s\}$ such that $j_k \notin \{i_1, ..., i_r\}$ thus

$$\mathbb{E}[Y_{i_1}^{k_1} \dots Y_{i_r}^{k_r} Y_{j_1}^{t_1} \dots Y_{j_s}^{t_s}] = 0.$$

□

2.4 Obtuse random walks

Let us recall briefly the canonical construction of discrete-time normal martingales with values in $\mathbb{R}^d$. Consider a normal martingale $(Y_0 + \ldots + Y_n)_{n\geq 0}$ such that, for each n, Y_n takes $d+1$ values $v_0(n), ..., v_d(n)$ conditionally to $\mathcal{F}_{n-1}$. Let $\mathbb{P}$ be any probability measure on the set $\Omega = \{0, ..., d\}^{\mathbb{N}}$ that assigns strictly positive probability p_n^i to each $v_i(n)$ where $(v_i(n))_n$ and $(p_n^i)_n$ are predictable processes. $(\mathcal{F}_n)_{n\geq 0}$ denote the filtration generated by $(Y_n)_{n\in\mathbb{N}}$ i.e.

$$\mathcal{F}_n = \sigma(Y_0, ..., Y_n), \quad n \in \mathbb{N}.$$

We introduce the coordinate maps

$$\begin{array}{rccc} X_n : & \Omega & \longrightarrow & \{0, 1, ..., d\} \\ & w & \longmapsto & w_n \end{array}$$

For

$$w = (w_0, w_1, ..., w_n, ...) \in \Omega,$$

we write $Y_n(w) = v_{X_n(w)}$ which yields

$$\mathcal{F}_n = \sigma(X_0, ..., X_n), \quad n \in \mathbb{N}.$$

Hence we have

$$p_n^i = \mathbb{P}(Y_n = v_i(n)\,|\mathcal{F}_{n-1}) = \mathbb{P}(X_n = i\,|\mathcal{F}_{n-1}), \quad n \in \mathbb{N}.$$

Let

$$c_i^j(n) = p_n^i v_i^j(n), \quad n \in \mathbb{N},\ i \in \{0, ..., d\} \text{ and } j \in \{1, ..., d\}.$$

Proposition 2.4.1. *$\forall n \in \mathbb{N}$, $\forall j, l \in \{1, ..., d\}$, we have*

$$\sum_{i=0}^{d} c_i^j(n) = 0,$$

and

$$\sum_{i=0}^{d} c_i^j(n) v_i^l(n) = \delta^{jl}.$$

Proof. We write

$$\sum_{i=0}^{d} c_i^j(n) = \mathbb{E}[Y_n^j \,|\mathcal{F}_{n-1}] = 0,$$

and

$$\sum_{i=0}^{d} c_i^j(n) v_i^l(n) = \mathbb{E}[Y_n^j Y_n^l \,|\mathcal{F}_{n-1}] = \delta^{jl}.$$

□

Recall that the filtration $(\mathcal{F}_n)_{n\geq 0}$ is said to be of multiplicity $d+1$ if each $\mathcal{F}_n$ is finite and each atom of $\mathcal{F}_n$ contains exactly $d+1$ atoms of $\mathcal{F}_{n+1}$. The following result gives a characterization of normal martingales which satisfy the CRP (c.f. [2] for a proof and for more details).

Theorem 2.4.2. *Let $(Y_0 + \ldots + Y_n)_{n\geq 0}$ be a d-dimensional normal martingale, the following assertions are equivalent*

1. *The filtration multiplicity is bounded from above by $d+1$.*
2. *The filtration multiplicity is exactly $d+1$.*
3. *$(Y_0 + \ldots + Y_n)_{n\geq 0}$ satisfies a SE*

$$Y_n^i Y_n^j = \delta^{ij} + \sum_{k=1}^{d} \Phi_{ij}^k(n) Y_n^k,$$

where Φ_{ij}^k are d^3 predictable processes.

4. *$(Y_0 + \ldots + Y_n)_{n\geq 0}$ has the PRP.*
5. *$(Y_0 + \ldots + Y_n)_{n\geq 0}$ has the CRP.*

Definition 2.4.3. *An obtuse random walk is a process that satisfies the equivalent condition of Theorem 2.4.2.*

Note that the values $(v_i(n))_{0\leq i\leq d}$ of Y_n and their probabilities $(p_n^i)_{0\leq i\leq d}$ are related to the coefficients of the SE by

$$\Phi(n) = \sum_{i=0}^{d} p_n^i v_i^*(n) \otimes v_i(n) \otimes v_i(n).$$

2.5 Chaotic representation property

Assume now that the filtration $(\mathcal{F}_n)_n$ generated by $(Y_n)_{n\in\mathbb{N}}$ has a multiplicity equal to $d+1$. Let $L^0(\Omega,\mathcal{F}_n)$ the space of $\mathcal{F}_n$-measurable random variables, it has finite dimension equal to $(d+1)^{n+1}$.
For $N\in\mathbb{N}$, we denote

$$\mathcal{I}_N^r(f_r) = \mathcal{I}^r\left(f_r 1_{[\![0,N]\!]^r}\right).$$

Note that if $r > N+1$, then $\mathcal{I}_N^r(f_r) = 0$.

Proposition 2.5.1. *For all $r \geq 1$,*

$$\mathcal{I}_N^r(f_r) = \mathbb{E}\left[\mathcal{I}^r(f_r)\,|\mathcal{F}_N\right].$$

Proof. Let $0 \leq i_1 < ... < i_r \ \in \Delta_r$, if $i_r > N$ we have

$$\mathbb{E}[Y_{i_1}^{k_1}...Y_{i_r}^{k_r}] = \mathbb{E}[\mathbb{E}[Y_{i_1}^{k_1}...Y_{i_r}^{k_r}]\,|\mathcal{F}_{r-1}] = 0.$$

As a result

$$\mathbb{E}[\mathcal{I}^r(f_r)] = 0, \quad \forall r \geq 1$$

and the process $(\mathcal{I}_k^r(f_r))_{k\in\mathbb{N}}$ is a discrete-time martingale. □

Corollary 2.5.2. *For $0 \leq N \leq r$,*
$\mathcal{I}^r(f_r)$ is $\mathcal{F}_N-$measurable if and only if $f_r 1_{[\![0,N]\!]^r} = f_r$.

Proof. The sufficiency is obvious. The necessity is a consequence of

$$\mathcal{I}_N^r(f_r) = \mathbb{E}\left[\mathcal{I}^r(f_r)\,|\mathcal{F}_N\right] = \mathcal{I}^r(f_r),$$

and of the isometry formula. □

Definition 2.5.3. *Let $\mathcal{H}_0 = \mathbb{R}$ and for $n \geq 1$, we denote $\mathcal{H}_n$ the subspace of $L^2(\Omega)$ made of stochastic integrals of order $n \geq 1$*

$$\mathcal{H}_n = \{\mathcal{I}^n(f_n), f_n \in L^2(\Delta_n, \mathbb{R}^{d^n})\}.$$

Proposition 2.5.4. $\forall n \in \mathbb{N}$,

$$L^0(\Omega,\mathcal{F}_n) \subset \mathcal{H}_0 \oplus ... \oplus \mathcal{H}_{n+1}.$$

Proof. For $0 \leq r \leq n+1$, we have $\dim L^0(\Omega,\mathcal{F}_n)\cap\mathcal{H}_r = \binom{n+1}{r}d^r$. More precisely,

$$\{Y_{i_1}^{k_1}...Y_{i_r}^{k_r} : \ 0 \leq i_1 < ... < i_r \leq n, \ 1 \leq k_1, ..., k_r \leq d\}$$

form an orthonormal basis. By orthogonality of the subspaces $\mathcal{H}_r$ we have

$$L^0(\Omega,\mathcal{F}_n) = (\mathcal{H}_0 \oplus ... \oplus \mathcal{H}_{n+1}) \cap L^0(\Omega,\mathcal{F}_n).$$

□

Consequence 2.5.5. *Any element $F \in L^2(\Omega, \mathcal{F}_n)$ can be written as*

$$F = \mathbb{E}[F] + \sum_{r=1}^{n+1} \mathcal{I}_n^r(f_r).$$

Definition 2.5.6. *We denote $\mathcal{S}$ the linear space spanned by multiple stochastic integrals*

$$\mathcal{S} = \left\{ \bigcup_{n=0}^{\infty} \mathcal{H}_n \right\} = \left\{ \sum_{r=0}^{n} \mathcal{I}^r(f_r), \text{with } f_r \in L^2(\Delta_r, \mathbb{R}^{d^r}) \text{ symmetric} \right\}.$$

The completion of $\mathcal{S}$ in $L^2(\Omega)$ is denoted by the direct sum

$$\bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

The next result establishes the CRP for normal martingales under the assumption that we move from $\mathcal{F}_n$ to $\mathcal{F}_{n+1}$ by decomposition of each atom of $\mathcal{F}_n$ to $d+1$ atoms (with measure > 0), see [2, 31].

Theorem 2.5.7.

$$L^2(\Omega) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n.$$

Proof. It suffices to show that $\mathcal{S}$ is dense in $L^2(\Omega)$. To this end, let F be a bounded random variable, then Proposition 2.5.4 shows that, $\mathbb{E}[F \,|\mathcal{F}_n] \in \mathcal{S}$. But, the martingale $(\mathbb{E}[F \,|\mathcal{F}_n])_{n\in\mathbb{N}}$ converges a.s. and in $L^2(\Omega)$ to F, and we are done. $\square$

2.6 Gradient operator

Definition 2.6.1. *The gradient operator $D : \mathcal{S} \longrightarrow L^2(\Omega \times \mathbb{N}, \mathbb{R}^d)$ is defined by*

$$D_k^j(\mathcal{I}^r(f_r)) = r\mathcal{I}^{r-1}\left(f_r^j(*, k)1_{\Delta_r}(*, k)\right), \qquad k \in \mathbb{N} \text{ and } j \in \{1, ..., d\}.$$

Proposition 2.6.2. *The gradient operator is continuous on the chaos $\mathcal{H}_r$.*

Proof. We have

$$\begin{aligned}
\|D_k\mathcal{I}^r(f_r)\|^2_{L^2(\Omega,\mathbb{R}^d)} &= \sum_{j=1}^{d} \|D_k^j\mathcal{I}^r(f_r)\|^2_{L^2(\Omega,\mathbb{R})} \\
&= \sum_{j=1}^{d} r^2\|\mathcal{I}^{r-1}\left(f_r^j(*, k)1_{\Delta_r}(*, k)\right)\|^2 \\
&= \sum_{j=1}^{d} r^2(r-1)!\|f_r^j(*, k)\|^2_{L^2(\Delta^{r-1})} \\
&= rr!\|f_r(*, k)\|^2_{L^2(\Delta^{r-1})}.
\end{aligned}$$

$\square$

Proposition 2.6.3. *Let $F \in \mathcal{S}$ be $\mathcal{F}_n$-measurable, then for any $k > n$, one has*

$$D_k^j F = 0, \quad j \in \{1, ..., d\}.$$

Proof. We write

$$F = \mathbb{E}[F] + \sum_{r=1}^{n+1} \mathcal{I}_n^r(f_r)$$

Then for $k > n$ we have

$$D_k^j \left(\mathcal{I}_n^r(f_r)\right) = r\mathcal{I}^{r-1}(f_r^j(*, k)1_{[0,n]^r}^j(*, k)1_{\Delta_r}(*, k)) = 0, \quad \forall j \in \{1, ..., d\}.$$

$\square$

Remark 2.6.4. *By the Clark-Ocone formula derived in the next section, the converse of this proposition is also true i.e. if $F \in \mathcal{S}$ is such that*

$$D_k^j F = 0, \ \forall k > n \ \text{and} \ \forall j \in \{1, ..., d\},$$

then F is $\mathcal{F}_n$-measurable.

Notation 2.6.5. *Let $\widetilde{f}_n \in L^2(\mathbb{R}^n, \mathbb{R}^{d^n})$ denote the symmetrization of $f_n \in L^2(\Delta_n, \mathbb{R}^{d^n})$, given by*

$$\widetilde{f}_n^{i_1,...,i_n}(t_1, ..., t_n) = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} f_n^{i_{\sigma(1)},...,i_{\sigma(n)}}(t_{\sigma(1)}, ..., t_{\sigma(n)}), \quad 1 \leq i_1, ..., i_n \leq d.$$

In particular, for $(s_1, ..., s_r) \in \Delta_r$, we have

$$\widetilde{1}_{\{s_1,...,s_r\}}^{i_1,...,i_r}(t_1, ..., t_r) = \frac{1}{r!} 1_{\{\{s_1,...,s_r\}=\{t_1,...,t_r\}\}} e_{i_1} \otimes ... \otimes e_{i_r},$$

where $(e_1, ..., e_d)$ denotes the canonical basis of $\mathbb{R}^d$.

Proposition 2.6.6. *For any $r \geq 1$, we have*

$$\mathcal{I}^r \left(\widetilde{1}_{(s_1,...,s_r)}^{i_1,...,i_r}\right) = Y_{s_1}^{i_1} ... Y_{s_r}^{i_r}.$$

As a result an orthonormal basis of $L^2(\Omega, \mathcal{F}_n)$ is given by

$$\left\{Y_{s_1}^{i_1} ... Y_{s_r}^{i_r} : \ 0 \leq s_1 < ... < s_r \leq n, \ 1 \leq i_1, ..., i_r \leq d\right\}.$$

Hence we can write

Proposition 2.6.7. *For any $r \geq 1$,*

$$D_k^j \left(Y_{s_1}^{i_1} ... Y_{s_r}^{i_r} \right) = \begin{cases} \delta^{j i_t} Y_{s_1}^{i_1} ... \check{Y}_{s_t}^{i_t} ... Y_{s_r}^{i_r} & if \;\; k = s_t, \quad t \in \{1, ..., r\} \\ 0 & if \;\; k \notin (s_1, ... s_r) \end{cases} .$$

where $\check{Y}_{s_t}^{i_t}$ denotes that the factor $Y_{s_t}^{i_t}$ should be omitted in the product.

Proof. Using Proposition 2.6.6, one can see that

$$\begin{aligned} D_k^j \left(Y_{s_1}^{i_1} ... Y_{s_r}^{i_r} \right) &= D_k^j \left(\mathcal{I}^r \left(\widetilde{1}_{(s_1, ..., s_r)}^{i_1, ..., i_r} \right) \right) \\ &= r \mathcal{I}^{r-1} \left(\frac{1}{r!} 1_{\{j \in (i_1, ..., i_r)\}} 1_{\{(s_1, ..., s_r) = (*, k)\}} 1_{\Delta_r}(*, k) e_{i_1} \otimes ... \otimes e_{i_{r-1}} \otimes e_j \right) \end{aligned}$$

□

The following result gives the probabilistic interpretation of D_k^j as a finite difference operator in the case of discrete time random walks with i.i.d. increments.

Proposition 2.6.8. *For any $F \in \mathcal{S}$, one has*

$$D_k^j F(w) = \sum_{i=0}^{d} c_i^j(k) F(w_i^k)$$

where we recall that (c.f. Section 4)

$$c_i^j(k) = \mathbb{P}(X_k = i \,|\, \mathcal{F}_{k-1}) = Y_k^j(w_i^k)$$

and

$$w_i^k = (w_1, ..., w_{k-1}, i, w_{k+1}, ...).$$

Proof. It suffices to consider $F = Y_{s_1}^{i_1} ... Y_{s_r}^{i_r}$. By Proposition 2.6.7

$$D_k^j F = \begin{cases} \delta^{j i_t} Y_{s_1}^{i_1} ... \check{Y}_{s_t}^{i_t} ... Y_{s_r}^{i_r} & if \;\; k = s_t, \quad t \in \{1, ..., r\} \\ 0 & if \;\; k \notin (s_1, ..., s_r) \end{cases} .$$

If $k \notin (s_1, ..., s_r)$, we get $F(w_i^k) = F(w)$ hence by Proposition 2.4.1,

$$\sum_{i=0}^{d} c_i^j(k) F(w_i^k) = \sum_{i=0}^{d} c_i^j(k) F(w) = 0 = D_k^j F(w)$$

Suppose now that $k \in (s_1, ..., s_r)$ for example, let $k = s_r$ then

$$D_k^j F = \delta^{j i_r} \prod_{p=1}^{r-1} Y_{s_p}^{i_p}.$$

But,

$$\begin{aligned}
\sum_{i=0}^{d} c_i^j(k)F(w_i^k) &= \sum_{i=0}^{d} c_i^j(k)\prod_{p=1}^{r} Y_{s_p}^{i_p}(w_i^k) \\
&= \left(\prod_{p=1}^{r-1} Y_{s_p}^{i_p}(w)\right)\sum_{i=0}^{d} c_i^j(k)Y_k^{i_r}(w_i^k) \\
&= \left(\prod_{p=1}^{r-1} Y_{s_p}^{i_p}(w)\right)\sum_{i=0}^{d} c_i^j(k)v_i^{i_r}(k) \\
&= \delta^{ji_r}\prod_{p=1}^{r-1} Y_{s_p}^{i_p}(w) \\
&= D_k^j F(w).
\end{aligned}$$

□

An immediate consequence, is the following

Corollary 2.6.9. *The gradient operator extends to any random variable $F : \Omega \longrightarrow \mathbb{R}$.*

We denote $Dom(D)$ the L^2-domain of D : $F \in Dom(D)$ if and only if

$$\mathbb{E}[\|DF\|^2_{l^2(\mathbb{N})}] < \infty.$$

2.7 Clark-Ocone formula

In this section, we derive an explicit expression for the predictable representation of stochastic variables. The main tool is a discrete time analog of the well-known Clark-Ocone formula : for any random variable F

$$F = \mathbb{E}[F] + \int \mathbb{E}[D_t F \,|\mathcal{F}_t] dB_t.$$

When $d = 1$, the discrete time analog of the Clark-Ocone formula for Bernoulli measures appears in [35] (we refer also to [27] for a discrete but finite Clark-Ocone formula). For general $d \geq 1$, one has

Proposition 2.7.1. *For any $F \in \mathcal{S}$,*

$$\begin{aligned}
F &= \mathbb{E}[F] + \sum_{k=0}^{\infty} < \mathbb{E}[D_k F \,|\mathcal{F}_{k-1}], Y_k > \\
&= \mathbb{E}[F] + \sum_{k=0}^{\infty} < D_k \mathbb{E}[F \,|\mathcal{F}_k], Y_k > .
\end{aligned}$$

Proof. By linearity, it suffices to show the result for $F = \mathcal{I}^r(f_r)$, from the recurrence formula

$$\begin{aligned} F &= r\sum_{i=1}^{d}\sum_{k=0}^{\infty}\mathcal{I}^{r-1}\left(f_r^i(*,k)1_{\Delta_r}(*,k)1_{[\![0,k-1]\!]^{r-1}}(*)\right)Y_k^i \\ &= r\sum_{i=1}^{d}\sum_{k=0}^{\infty}\mathbb{E}\left[\mathcal{I}^{r-1}\left(f_r^i(*,k)1_{\Delta_r}(*,k)\right)\,|\mathcal{F}_{k-1}\right]Y_k^i \\ &= \sum_{i=1}^{d}\sum_{k=0}^{\infty}\mathbb{E}[D_k^i F\,|\mathcal{F}_{k-1}]Y_k^i \end{aligned}$$

and since $\mathbb{E}[\mathcal{I}^r(f_r)] = 0$, $\forall r \geq 1$ we get the first identity, while the second one holds from

$$\mathbb{E}[D_k^i F\,|\mathcal{F}_{k-1}] = D_k^i\mathbb{E}[F\,|\mathcal{F}_k].$$

□

Proposition 2.7.2. *The operator*

$$\begin{array}{ccc} L^2(\Omega) & \longrightarrow & L^2(\Omega\times\mathbb{N},\mathbb{R}^d) \\ F & \longmapsto & ((\mathbb{E}[D_k^1 F\,|\mathcal{F}_{k-1}],...,\mathbb{E}[D_k^d F\,|\mathcal{F}_{k-1}]))_{k\in\mathbb{N}} \end{array}$$

is bounded with norm equal to one, hence the Clark-Ocone formula extends to any $F \in L^2(\Omega)$.

Proof. From the Clark-Ocone formula and using the isometry formula, we have for $F \in \mathcal{S}$;

$$\begin{aligned} \|\mathbb{E}[D_. F\,|\mathcal{F}_{.-1}]\|^2_{L^2(\Omega\times\mathbb{N})} &= \sum_{j=1}^{d}\|\mathbb{E}[D_.^j F\,|\mathcal{F}_{.-1}]\|^2_{L^2(\Omega\times\mathbb{N})} \\ &= \sum_{j=1}^{d}\mathbb{E}\left[\sum_{k=0}^{\infty}\left(\mathbb{E}[D_k^j F\,|\mathcal{F}_{k-1}]\right)^2\right] \\ &= \mathbb{E}[(F-\mathbb{E}[F])^2] \\ &\leq \mathbb{E}[|F|^2] \\ &= \|F\|^2_{L^2(\Omega)}. \end{aligned}$$

□

Consequently we state a Poincaré inequality.

Corollary 2.7.3. *For any $F \in L^2(\Omega)$,*

$$var(F) \leq \|DF\|^2_{L^2(\Omega\times\mathbb{N},\mathbb{R}^d)}.$$

Proof. We have

$$\begin{aligned}
var(F) &= \mathbb{E}[|F-\mathbb{E}[F]|^2] \\
&= \sum_{j=1}^{d}\mathbb{E}\left[\left(\sum_{k=0}^{\infty}\mathbb{E}[D_k^j F\,|\mathcal{F}_{k-1}]\right)^2\right] \\
&= \sum_{j=1}^{d}\mathbb{E}\left[\sum_{k=0}^{\infty}\left(\mathbb{E}[D_k^j F\,|\mathcal{F}_{k-1}]\right)^2\right] \\
&\leq \sum_{j=1}^{d}\mathbb{E}\left[\sum_{k=0}^{\infty}\mathbb{E}[|D_k^j F|^2\,|\mathcal{F}_{k-1}]\right] \\
&= \sum_{j=1}^{d}\mathbb{E}\left[\sum_{k=0}^{\infty}|D_k^j F|^2\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty}\|D_k F\|^2\right].
\end{aligned}$$

□

Another variant of the Clark-Ocone formula is stated as

Proposition 2.7.4. *For $n\in\mathbb{N}$ and $F\in L^2(\Omega)$. We have*

$$F=\mathbb{E}[F|\,\mathcal{F}_n]+\sum_{k=n+1}^{\infty}<\mathbb{E}[D_kF\,|\mathcal{F}_{k-1}],Y_k>$$

and

$$\mathbb{E}[F^2]=\mathbb{E}[(\mathbb{E}[F\,|\mathcal{F}_n])^2]+\mathbb{E}\left[\sum_{k=n+1}^{\infty}\|\mathbb{E}[D_kF\,|\mathcal{F}_{k-1}]\|^2\right].$$

Proof. Since $\mathbb{E}[F\,|\mathcal{F}_n]\in L^2(\Omega,\mathcal{F}_n)$, the Clark-Ocone formula gives

$$\begin{aligned}
\mathbb{E}[F\,|\mathcal{F}_n] &= \mathbb{E}[F]+\sum_{i=0}^{d}\sum_{k=0}^{n}\mathbb{E}[D_k^j\mathbb{E}[F\,|\mathcal{F}_n]\,|\mathcal{F}_{k-1}]Y_k^j \\
&= \mathbb{E}[F]+\sum_{i=1}^{d}\sum_{k=0}^{n}D_k^j\mathbb{E}[\mathbb{E}[F\,|\mathcal{F}_n]\,|\mathcal{F}_k]Y_k^j \\
&= \mathbb{E}[F]+\sum_{i=1}^{d}\sum_{k=0}^{n}D_k^j\mathbb{E}[F\,|\mathcal{F}_k]Y_k^j \\
&= \mathbb{E}[F]+\sum_{i=1}^{d}\sum_{k=0}^{n}\mathbb{E}[D_k^jF\,|\mathcal{F}_{k-1}]Y_k^j
\end{aligned}$$

which proves the first identity. The second identity is a consequence of the first one together with the isometry property of $\mathcal{I}$. $\square$

As an application of the Clark-Ocone formula, we obtain the following PRP for discrete-time martingales.

Proposition 2.7.5. *Let $(M_n)_{n\in\mathbb{N}}$ be a d-dimensional martingale in $L^2(\Omega)$ with respect to $(\mathcal{F}_n)_{n\in\mathbb{N}}$. There exists a predictable d-dimensional process $(\gamma_k)_{k\in\mathbb{N}}$ such that $\forall n \in \mathbb{N}$,*

$$M_n^i = M_{-1}^i + \sum_{k=0}^{n} < \gamma_k^i, Y_k >, \qquad i \in \{1, ..., d\}.$$

Proof. Let $k \geq 1$. The Corollary 2.7.4 shows that $\forall i \in \{1, ..., d\}$

$$\begin{aligned} M_k^i &= \mathbb{E}[M_k^i \,|\mathcal{F}_{k-1}] + < \mathbb{E}[D_k M_k^i \,|\mathcal{F}_{k-1}], Y_k > \\ &= M_{k-1}^i + < \mathbb{E}[D_k M_k^i \,|\mathcal{F}_{k-1}], Y_k > \end{aligned}$$

and one concludes by letting

$$\gamma_k^i := \mathbb{E}[D_k M_k^i \,|\mathcal{F}_{k-1}], \quad k \geq 0.$$

$\square$

2.8 Divergence operator

Let $\mathcal{U}$ the subspace of $L^2(\Omega \times \mathbb{N}, \mathbb{R}^d)$ defined by

$$\mathcal{U} = \Big\{ \sum_{r=0}^{n} \left(\mathcal{I}^r\left(f_{r+1}^1(*, .)\right), ..., \mathcal{I}^r\left(f_{r+1}^d(*, .)\right)\right), \\ f_{r+1} \in L^2(\Delta_{r+1}) \text{ symmetric on its first } r \text{ variables} \Big\}$$

Definition 2.8.1. *The divergence operator is the linear mapping $\delta : \mathcal{U} \longrightarrow L^2(\Omega)$ defined by*

$$\delta(X) = \delta\left(\mathcal{I}^r\left(f_{r+1}^1(*, .)\right), ..., \mathcal{I}^r\left(f_{r+1}^d(*, .)\right)\right) = \mathcal{I}^{r+1}\left(\widetilde{f}_{r+1}\right)$$

for $(X_k)_k$ of the form

$$X_k = \left(\mathcal{I}^r\left(f_{r+1}^1(*, k)\right), ..., \mathcal{I}^r\left(f_{r+1}^d(*, k)\right)\right), \quad k \in \mathbb{N}.$$

Proposition 2.8.2. *The operator δ is the adjoint to D.*

Proof. We consider $F = \mathcal{I}^r(f_r)$ and $G = (G_k = (G_k^1, ..., G_k^d))_{k\in\mathbb{N}}$ where

$$G_k^i = \mathcal{I}^s(g_{s+1}^i(*, k)), \forall 1 \geq i \geq d.$$

We have

$$\begin{aligned}
&\mathbb{E}\left[< D_. F, G >_{l^2(\mathbb{N})}\right] \\
&= \sum_{i=1}^{d} \mathbb{E}\left[< D_.^i F, G^i >_{l^2(\mathbb{N})}\right] \\
&= \sum_{i=1}^{d} r \sum_{k=0}^{\infty} \mathbb{E}\left[\mathcal{I}^{r-1}(f_r^i(*, k) 1_{\Delta_r}(*, k)) \mathcal{I}^s(g_{s+1}^i(*, k))\right] \\
&= r! 1_{\{r-1=s\}} \sum_{i=1}^{d} \sum_{\substack{i_1,\dots,i_{r-1}=1 \\ j_1,\dots,j_s=1}}^{d} < f_r^{i_1,\dots,i_{r-1},i}, g_r^{j_1,\dots,j_s,i} > 1_{\{i_1=j_1,\dots i_{r-1}=j_s\}} \\
&= \mathbb{E}\left[\mathcal{I}^r(f_r) \mathcal{I}^{s+1}(\widetilde{g}_{s+1})\right] \\
&= \mathbb{E}\left[\delta(G) F\right]
\end{aligned}$$

□

The following result shows that δ coincides with the stochastic integral operator $\mathcal{I}$ on the square summable predictable processes.

Proposition 2.8.3. *The operator δ can be extended to $L^2(\Omega \times \mathbb{N}, \mathbb{R}^d)$ with*

$$\delta(X) = \sum_{k=0}^{\infty} < X_k, Y_k > - \sum_{i=1}^{d} \sum_{k=0}^{\infty} < D_k(X_k^i), Y_k > Y_k^i, \tag{2.8.1}$$

provided all series converge in $L^2(\Omega)$.

Proof. Let $X_k = \left(\mathcal{I}^r\left(f^1_{r+1}(*,k)\right), ..., \mathcal{I}^r\left(f^d_{r+1}(*,k)\right)\right)$, then one has

$$\begin{aligned}
&\delta(X)\\
&=\mathcal{I}^{r+1}\left(\widetilde{f}_{r+1}\right)\\
&=\sum_{k_1,...,k_{r+1}=1}^{d}\ \sum_{(i_1,...,i_{r+1})\in\Delta_{r+1}} \widetilde{f}^{k_1,...,k_{r+1}}_{r+1}(i_1,...,i_{r+1})Y^{k_1}_{i_1}...Y^{k_{r+1}}_{i_{r+1}}\\
&=\sum_{k_1,...,k_r,t=1}^{d}\ \sum_{k=0}^{\infty}\ \sum_{(i_1,...,i_r)\in\Delta_r} \widetilde{f}^{k_1,...,k_r,t}_{r+1}(i_1,...,i_r,k)Y^{k_1}_{i_1}...Y^{k_r}_{i_r}Y^t_k\\
&-r\sum_{k_1,...,k_{r-1},s,t=1}^{d}\ \sum_{k=0}^{\infty}\ \sum_{(i_1,...,i_{r-1})\in\Delta_{r-1}} \widetilde{f}^{k_1,...,k_{r-1},s,t}_{r+1}(i_1,...,i_{r-1},k,k)Y^{k_1}_{i_1}...Y^{k_{r-1}}_{i_{r-1}}Y^s_kY^t_k\\
&=\sum_{k=0}^{\infty}\mathcal{I}^r\left(f^t_{r+1}(*,k)\right)Y^t_k - r\sum_{s,t=1}^{d}\ \sum_{k=0}^{\infty}\mathcal{I}^{r-1}\left(f^{s,t}_{r+1}(*,k,k)1_{\Delta_{r+1}}(*,k,k)\right)Y^s_kY^t_k\\
&=\sum_{k=0}^{\infty}<X_k,Y_k> - \sum_{s,t=1}^{d}\ \sum_{k=0}^{\infty}D^s_k(X^t_k)Y^s_kY^t_k.
\end{aligned}$$

□

Observe that from Proposition 2.6.3, the last term in the right-hand side of (2.8.1) vanish when X is predictable. As a result

Corollary 2.8.4. *δ coincides with the stochastic integral operator $\mathcal{I}$ on the square summable predictable process.*

2.9 Covariance identities

The covariance $Cov(F,G)$ of $F,G\in L^2(\Omega)$ is defined as

$$Cov(F,G) = \mathbb{E}[(F-\mathbb{E}[F])(G-\mathbb{E}[G])] = \mathbb{E}[FG]-\mathbb{E}[F]\mathbb{E}[G].$$

Let $(P_t)_{t\in\mathbb{R}_+}$ denote the Ornstein–Uhlenbeck semigroup, defined as

$$P_tF = \sum_{n=0}^{\infty} e^{-nt}\mathcal{I}^n(f_n)$$

where $F=\sum_{n=0}^{\infty}\mathcal{I}^n(f_n)$. We have

Proposition 2.9.1. *For $F, G \in Dom(D)$,*

$$Cov(F,G) = \mathbb{E}\left[\sum_{k=0}^{\infty} < \mathbb{E}[D_k F \,|\mathcal{F}_{k-1}], D_k G >\right]$$

and

$$Cov(F,G) = \mathbb{E}\left[\sum_{k=0}^{\infty} \int_0^{\infty} e^{-t} < D_k F, P_t D_k G > dt\right].$$

Proof. The first identity is a consequence of the Clark-Ocone formula :

$$\begin{aligned}
Cov(F,G) &= \mathbb{E}[(F-\mathbb{E}[F])(G-\mathbb{E}[G])] \\
&= \mathbb{E}\left[\left(\sum_{k=0}^{\infty} < \mathbb{E}[D_k F \,|\mathcal{F}_{k-1}], Y_k >\right)\left(\sum_{l=0}^{\infty} < \mathbb{E}[D_l G \,|\mathcal{F}_{l-1}], Y_l >\right)\right] \\
&= \mathbb{E}\left[\left(\sum_{k=0}^{\infty}\sum_{i=1}^{d} \mathbb{E}[D_k^i F \,|\mathcal{F}_{k-1}] Y_k^i\right)\left(\sum_{l=0}^{\infty}\sum_{i=1}^{d} \mathbb{E}[D_l^i G \,|\mathcal{F}_{l-1}] Y_l^i\right)\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty}\sum_{i=1}^{d} \mathbb{E}[D_k^i F \,|\mathcal{F}_{k-1}]\mathbb{E}[D_k^i G \,|\mathcal{F}_{k-1}]\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty}\sum_{i=1}^{d} \mathbb{E}\left[\mathbb{E}[D_k^i F \,|\mathcal{F}_{k-1}] D_k^i G \,|\mathcal{F}_{k-1}\right]\right] \\
&= \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{E}\left[< \mathbb{E}[D_k F \,|\mathcal{F}_{k-1}], D_k G > \,|\mathcal{F}_{k-1}\right]\right] \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} < \mathbb{E}[D_k F \,|\mathcal{F}_{k-1}], D_k G >\right].
\end{aligned}$$

By orthogonality of multiple integrals of different orders and continuity of P_t on $L^2(\Omega)$, it suffices to prove the second identity for $F = \mathcal{I}^n(f_n)$ and $G = \mathcal{I}^n(g_n)$. We have

$$\begin{aligned}
& Cov(\mathcal{I}^n(f_n), \mathcal{I}^n(g_n)) \\
&= \mathbb{E}[\mathcal{I}^n(f_n)\mathcal{I}^n(g_n)] \\
&= n! \sum_{k_1,\ldots,k_n=1}^{d} < f_n^{k_1,\ldots,k_n}, g_n^{k_1,\ldots,k_n} > \\
&= n!n \int_0^\infty e^{-nt} dt \sum_{k_1,\ldots,k_n=1}^{d} < f_n^{k_1,\ldots,k_n}, g_n^{k_1,\ldots,k_n} > \\
&= n \int_0^\infty e^{-nt} dt \sum_{k_1,\ldots,k_n=1}^{d} \sum_{(i_1,\ldots,i_n)\in\Delta_n} f_n^{k_1,\ldots,k_n}(i_1,\ldots,i_n) g_n^{k_1,\ldots,k_n}(i_1,\ldots,i_n) \\
&= n^2 \mathbb{E}\left[\int_0^\infty e^{-t} \sum_{j=1}^{d}\sum_{k=0}^{\infty} \mathcal{I}^{n-1}\left(f_n^j(*,k)1_{\Delta_n}(*,k)\right) e^{(n-1)t}\mathcal{I}^{n-1}\left(g_n^j(*,k)1_{\Delta_{n-1}}(*,k)\right) dt\right] \\
&= n^2 \mathbb{E}\left[\int_0^\infty e^{-t} \sum_{j=1}^{d}\sum_{k=0}^{\infty} \mathcal{I}^{n-1}\left(f_n^j(*,k)1_{\Delta_n}(*,k)\right) P_t\mathcal{I}^{n-1}\left(g_n^j(*,k)1_{\Delta_{n-1}}(*,k)\right) dt\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-t} \sum_{j=1}^{d}\sum_{k=0}^{\infty} D_k^j\mathcal{I}^n(f_n)\, P_t D_k^j \mathcal{I}^n(g_n)\, dt\right].
\end{aligned}$$

□

The next result shows that $(P_t)_{t\in\mathbb{R}_+}$ admits an integral representation by a probability kernel.

Let us define the probability kernel $Q_t(\widetilde{w}, dw)$, for any $N \geq 1$ and $t \in \mathbb{R}_+$, by

$$\mathbb{E}\left[\frac{dQ_t(\widetilde{w}, .)}{d\mathbb{P}}\middle| \mathcal{F}_N\right](w) = q_t^N(\widetilde{w}, w)$$

where $q_t^N : \Omega \times \Omega \to \mathbb{R}_+$ is defined by

$$q_t^N(\widetilde{w}, w) = \prod_{i=0}^{N}\left(1 + e^{-t} < Y_i(w), Y_i(\widetilde{w}) >\right), \quad w, \widetilde{w} \in \Omega.$$

Proposition 2.9.2. *For any $F \in L^2(\Omega, \mathcal{F}_N)$ one has*

$$P_t F(\widetilde{w}) = \int_\Omega F(w) Q_t(\widetilde{w}, dw), \quad \widetilde{w} \in \Omega.$$

Proof : Recall that $L^2(\Omega, \mathcal{F}_N)$ has finite dimension $(d+1)^{N+1}$. More precisely, an orthonormal basis of $L^2(\Omega, \mathcal{F}_N)$ is given by

$$\left\{ Y_{k_1}^{s_1} \dots Y_{k_n}^{s_n} : \ 0 \le k_1 < \dots < k_n \le N, \ 1 \le s_1, \dots, s_n \le d \right\}.$$

Then it suffices to consider functionals of the form $F = Y_{k_1}^{s_1} \dots Y_{k_n}^{s_n}$. Now observe that

$$\begin{aligned}
\mathbb{E}\left[Y_k^j(.)\left(1 + e^{-t} < Y_k(.), Y_k(w) >\right)\right] &= \mathbb{E}\left[Y_k^j(.)\left(1 + e^{-t}\sum_{l=1}^{d} Y_k^l(.) Y_k^l(w)\right)\right] \\
&= \sum_{i=0}^{d} c_i^j(k)(1 + e^{-t}\sum_{l=1}^{d} v_i^l(k) Y_k^l(w)) \\
&= e^{-t}\sum_{i=0}^{d} c_i^j(k)\sum_{l=1}^{d} v_i^l(k) Y_k^l(w) \\
&= e^{-t}\sum_{l=1}^{d} \delta^{jl} Y_k^l(w) \\
&= e^{-t} Y_k^j(w).
\end{aligned}$$

Then, by independence of the sequence $(X_k)_{k\ge 0}$,

$$\begin{aligned}
\mathbb{E}\left[Y_{k_1}^{s_1} \dots Y_{k_n}^{s_n} q_t^N(\widetilde{w}, .)\right] &= \mathbb{E}\left[Y_{k_1}^{s_1} \dots Y_{k_n}^{s_n}\prod_{i=0}^{N}\left(1 + e^{-t} < Y_{k_i}(\widetilde{w}), Y_{k_i}(.) >\right)\right] \\
&= \prod_{i=0}^{N}\mathbb{E}\left[Y_{k_i}^{s_i}\left(1 + e^{-t} < Y_{k_i}(\widetilde{w}), Y_{k_i}(.) >\right)\right] \\
&= e^{-nt} Y_{k_1}^{s_1}(w) \dots Y_{k_n}^{s_n}(w) \\
&= e^{-nt}\mathcal{I}^n\left(\widetilde{1}_{\{k_1,\dots,k_n\}}^{s_1,\dots,s_n}\right)(w) \\
&= P_t(Y_{k_1}^{s_1} \dots Y_{k_n}^{s_n}).
\end{aligned}$$

□

2.10 Deviation inequality

In this section, we prove a deviation inequality for functionals of obtuse random walks, using the action of gradient operator and the covariance representations instead of the logarithm Sobolev inequality. We refer to [35, 23] for other versions of this inequality in the one-dimensional case.

Proposition 2.10.1. *Let $F : \Omega \to \mathbb{R}$ be a bounded random variable such that for any $k \in \mathbb{N}$ and $i, i' \in \{0, ..., d\}$,*

$$|F(w_i^k) - F(w_{i'}^k)| \leq K, \quad |c_i^j(k)| \leq C, \ j \in \{1, ..., d\}$$

for some $K, C > 0$ and $||DF||_{L^\infty(\Omega, l^1(\mathbb{N}))} < \infty$. Then

$$\begin{aligned} \mathbb{P}(F - \mathbb{E}[F] \geq x) &\leq \exp\left(-\frac{dC||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}}{K} g\left(\frac{x}{dC||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}}\right)\right) \\ &\leq \exp\left(-\frac{x}{2K} \ln\left(1 + \frac{x}{dC||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}}\right)\right) \end{aligned}$$

with $g(u) = (1+u)\ln(1+u) - u, \ u \geq 0$.

Proof : For sake of simplicity, we assume that $\mathbb{E}[F] = 0$. Then from the Chebychev inequality, one sees that

$$\mathbb{P}(F \geq x) \leq e^{-tx}\mathbb{E}\left[e^{tF}\right].$$

Next, letting $L(t) = \mathbb{E}\left[e^{tF}\right]$, one has

$$\begin{aligned} \ln(\mathbb{E}\left[e^{tF}\right]) &= \int_0^t \frac{L'(s)}{L(s)} ds \\ &= \int_0^t \frac{\mathbb{E}\left[Fe^{sF}\right]}{\mathbb{E}\left[e^{sF}\right]} ds. \end{aligned}$$

Now, we shall need the following result.

Lemma 2.10.2. *For any random variable $F : \Omega \to \mathbb{R}$ and $s \geq 0$, one has*

$$e^{-sF} D_k^j e^{sF} = \sum_{i=0, i \neq X_k}^{d} c_i^j(k) \left(e^{s(F(w_i^k) - F)} - 1\right).$$

Proof : We have

$$\begin{aligned} D_k^j e^F &= \sum_{i=0}^{d} c_i^j(k) e^{F(w_i^k)} \\ &= \sum_{l=0}^{d} \mathbf{1}_{\{X_k = l\}} \sum_{i=0, i \neq l}^{d} c_i^j(k) e^{F(w_i^k)} + c_l^j(k) e^F \\ &= \sum_{l=0}^{d} \mathbf{1}_{\{X_k = l\}} \sum_{i=0, i \neq l}^{d} c_i^j(k) e^{F(w_i^k)} - \sum_{i=0, i \neq l}^{d} c_i^j(k) e^F \\ &= \sum_{l=0}^{d} \mathbf{1}_{\{X_k = l\}} e^F \sum_{i=0, i \neq l}^{d} c_i^j(k) \left(e^{F(w_i^k) - F} - 1\right). \end{aligned}$$

$\square$

Remark 2.10.3. *Note that the gradient operator does not satisfy a derivation rule for products. More precisely, for any $F, G : \Omega \to \mathbb{R}$, we have*

$$D_k^j(FG) = FD_k^j(G) + GD_k^j(F) + \sum_{i=0, i \neq X_k}^{d} c_i^j(k)(F - F(w_i^k))(G - G(w_i^k)).$$

Let $(P_t)_{t \in \mathbb{R}_+}$ be the Ornstein–Uhlenbeck semigroup, defined in the previous section. From Proposition 2.9.2, we obtain the following bound.

Lemma 2.10.4. *For any $u \in L^2(\Omega \times \mathbb{N})$, one has*

$$||P_t u||_{L^\infty(\Omega, l^1(\mathbb{N}))} \leq ||u||_{L^\infty(\Omega, l^1(\mathbb{N}))}$$

Proof : Using the representation formula of P_t given in Proposition 2.9.2, one has $\mathbb{P}(d\widetilde{w})$-a.s.

$$\begin{aligned}
||P_t u||_{l^1(\mathbb{N})}(\widetilde{w}) &= \sum_{k=0}^{\infty} |P_t u_k(\widetilde{w})| \\
&= \sum_{k=0}^{\infty} \left| \int_\Omega u_k(w) Q_t(\widetilde{w}, dw) \right| \\
&\leq \sum_{k=0}^{\infty} \int_\Omega |u_k(w)| Q_t(\widetilde{w}, dw) \\
&= \int_\Omega ||u||_{l^1(\mathbb{N})}(w) Q_t(\widetilde{w}, dw) \\
&\leq ||u||_{L^\infty(\Omega, l^1(\mathbb{N}))}.
\end{aligned}$$

$\square$

Since the function $x \mapsto e^x - 1$ is positive and increasing on $\mathbb{R}$, then by Lemma 2.10.2, we obtain

$$\begin{aligned}
e^{-sF} D_k^j e^{sF} &= \sum_{i=0, i \neq X_k}^{d} c_i^j(k) \left(e^{s(F(w_i^k) - F)} - 1 \right) \\
&\leq (e^{sK} - 1) \sum_{i=0, i \neq X_k}^{d} c_i^j(k) \\
&= -c_{X_k}^j(k)(e^{sK} - 1) \\
&\leq C(e^{sK} - 1).
\end{aligned}$$

Going back to the proof of Proposition 2.10.1, by Proposition 2.9.1 and Lemma 2.10.4 one has

$$\begin{aligned}
\mathbb{E}\left[Fe^{sF}\right] &= \mathrm{Cov}(F, e^{sF}) \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty}\int_0^{\infty} e^{-t} < D_k e^{sF}, P_t D_k F > dt\right] \\
&\leq dC(e^{sK}-1)\mathbb{E}\left[e^{sF}\int_0^{\infty} e^{-t}||P_t DF||_{(l^1(\mathbb{N}),\mathbb{R}^d)} dt\right] \\
&\leq dC(e^{sK}-1)\mathbb{E}\left[e^{sF}\right]||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}\int_0^{\infty} e^{-t}dt \\
&\leq dC(e^{sK}-1)\mathbb{E}\left[e^{sF}\right]||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}.
\end{aligned}$$

Consequently we have

$$\begin{aligned}
\ln\left(\mathbb{E}\left[e^{tF}\right]\right) &= \int_0^t \frac{\mathbb{E}\left[Fe^{sF}\right]}{\mathbb{E}\left[e^{sF}\right]} ds \\
&\leq dC||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}\int_0^t (e^{sK}-1)ds \\
&\leq dC||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}(e^{tK}-tk-1).
\end{aligned}$$

As a result, for any $x, t \geq 0$,

$$\mathbb{P}(F \geq x) \leq \exp\left(dC||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}(e^{tK}-tk-1)-tx\right).$$

The minimum in $t \geq 0$ in the above expression is attained with

$$t = \frac{1}{K}\ln\left(1+\frac{x}{dC||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}}\right),$$

whence

$$\begin{aligned}
&\mathbb{P}(F \geq x) \\
&\leq \exp\left(-\frac{1}{K}((x+dC||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))})\ln\left(1+\frac{x}{dC||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}}\right)-x)\right) \\
&\leq \exp\left(-\frac{x}{2K}\ln\left(1+\frac{x}{dC||DF||_{L^\infty(\Omega,l^1(\mathbb{N}))}}\right)\right),
\end{aligned}$$

where we used the inequality $(1+u)\ln(1+u)-u \geq \frac{u}{2}\ln(1+u)$. $\square$

2.11 Complete markets in discrete time

In this section we present a complete market model in discrete time as an application of the Clark-Ocone formula. The discrete-time and finite horizon models of financial markets can be described as follows. We consider a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_n)_{-1\leq n\leq N}, \mathbb{P})$ where N is finite, $\mathcal{F}_{-1} = \{\varnothing, \Omega\}$ and $\mathcal{F} = \mathcal{F}_N$. Let

$$\widetilde{S} = (B, S^1, ..., S^d) = (B, S) \in \mathbb{R}^{d+1}$$

be $d+1$ assets such that $B = (B_n)_n$ is the price process of a risk-less asset (bond) where $B_n > 0$, $\forall n \geq -1$ and $S = (S^1, ..., S^d)$ is the price process assets (stocks). In order to distinguish the scalar product in $\mathbb{R}^{d+1}$, it will be convenient to use the notation $x.y = \sum_{i=1}^{d+1} x_i y_i$ for $x, y \in \mathbb{R}^{d+1}$. We start by giving some classical definitions for financial market.

Definition 2.11.1. *A portfolio (or strategy) $\pi \in \mathbb{R}^{d+1}$ is a pair (β, γ) , where $\beta = (\beta_n)$ and $\gamma_n = (\gamma_n^1, ..., \gamma_n^d)$ are predictable processes such that γ_n^i denote the number of shares of the i^{th} stock and β_n the number of bonds that the seller of the option owns at time n.*

The corresponding value at time n of the seller's portfolio is defined by

$$V_n^\pi = \pi_{n+1}.\widetilde{S}_n = \beta_{n+1}B_n + < \gamma_{n+1}, S_n > \qquad n \geq -1. \tag{2.11.2}$$

Note that in order to be consistent with the notation of the previous sections, we use the time scale $\mathbb{N}$, hence the index 0 is that of the first value of any stochastic process, while the index -1 corresponds to its deterministic initial value.

Definition 2.11.2. *A portfolio $\pi = (\beta, \gamma)$ is said to be self-financing if*

$$B_n \Delta\beta_n + < S_n, \Delta\gamma_n >= 0, \qquad n \geq 0, \tag{2.11.3}$$

where $\Delta Y_n = Y_{n+1} - Y_n$ denotes the increment of the process Y at time n.

Hence the self-financing condition implies

$$V_n^\pi = \beta_n B_n + < \gamma_n, S_n > .$$

It's also convenient to use the discounted prices $\overline{S} = (\overline{S}^1, ..., \overline{S}^d)$ defined by

$$\overline{S}_n = \frac{1}{B_n} S_n, \quad n \geq -1,$$

and the corresponding discounted value process of a strategy π defined as

$$\overline{V}_n^\pi = \frac{1}{B_n} V_n^\pi, \quad n \geq -1.$$

Definition 2.11.3. *A model is said to be arbitrage-free if for every self-financing* π *with* $V_0^\pi = 0$ *and* $V_N^\pi \geq 0$ *a.s. then* $V_N^\pi = 0$ *a.s.*

Definition 2.11.4. *An arbitrage-free market model is called complete if every contingent claim is attainable, i.e. every bounded* $\mathcal{F}$*-measurable random variable* F *can be hedged by a self-financing strategy.*

Note that in discrete-time setting, only a very limited class of models enjoys the completeness property. Let us recall the two basic theorems in asset pricing theory (see e.g. [10, 19, 25] for proofs and more details).

Definition 2.11.5. *An equivalent martingale measure* $\mathbb{Q}$ *is a probability measure equivalent to* $\mathbb{P}$ *under which the d-dimensional discounted process* $\overline{S}$ *is a martingale.*

Theorem 2.11.6. *A market model is arbitrage-free if and only if there exists an equivalent martingale measure.*

Theorem 2.11.7. *An arbitrage-free market model is complete if and only if there exists exactly one equivalent martingale measure. In this case, the number of atoms in* $(\Omega, \mathcal{F}, \mathbb{P})$ *is bounded above by* $(d+1)^N$.

This theorem suggests that the price process of a complete market model,can be constructed from an obtuse random walk. Throughout the following, we are interested in market models given by

$$\forall n \in \{0, ..., N\}, \qquad \begin{cases} S_n = \prod_{k=0}^n (I + M_k^i) S_{-1} & \text{if } X_n = i, \quad i \in \{0, ..., l\} \\ B_n = \prod_{k=0}^n (1 + r_k) \end{cases}$$

with initial values S_{-1} and $B_{-1} = 1$, where X_n is the coordinate maps i.e.

$$\begin{array}{rccc} X_n : & \Omega & \longrightarrow & \{0, ..., l\} \\ & w = (w_0, ..., w_N) & \longmapsto & w_n \end{array}$$

and $(M_k^i)_k$, $(r_k)_k$ are deterministic sequences such that $I + M_k^i$ is a matrix with non-negative entries and $r_k > -1$.

2.11.1 One-period model

We start by discussing the notions of arbitrage-freeness and completeness of the market in one-period model i.e. the assets are priced at the initial time $t = 0$ and at the final time $t = 1$. Let

$$\widetilde{\xi} = (1, \xi) = (1, \xi^1, ..., \xi^d) \in \mathbb{R}_+^{d+1}$$

and

$$\widetilde{S}^i = (1, S^i) = (1 + r, (I + M^i)\xi), \quad if \ X = i \in 0, ..., l,$$

the respective asset price at time $t = 0$ and $t = 1$. Let us consider

$$\pi = (\pi^0, \gamma) = (\pi^0, \gamma^1, ..., \gamma^d) \in \mathbb{R}^{d+1}$$

a portfolio at $t = 0$. The price for buying π equals $\pi.\overline{\xi}$ and at time $t = 1$, the portfolio takes the value

$$\pi.\widetilde{S}^i = (1 + r)\pi^0 + < (I + M^i)\xi, \gamma >,$$

depending on the scenario $X = i$.

With these notations, Theorem 2.11.6 implies that the arbitrage-freeness is equivalent to the existence of an equivalent martingale measure $\mathbb{Q}$ such that the probabilities $q_i = \mathbb{Q}(X = i) > 0$ solve the linear equations

$$\begin{cases} q_0 S^0 + ... + q_l S^l = r\xi \\ q_0 + ... + q_l = 1. \end{cases}$$

If a solution exists, it will be unique (i.e. the arbitrage-free market model is complete) if and only if $l = d$ and

$$\begin{pmatrix} S^0 \\ 1 \end{pmatrix}, ..., \begin{pmatrix} S^d \\ 1 \end{pmatrix}$$

are linearly independent in $\mathbb{R}^{d+1}$.

Remark 2.11.8. *Note that if the arbitrage-free market model is complete, then the collection $\{S^0, ..., S^d\}$ generates a convex set which contains the origin (c.f. [19]).*

2.11.2 Multi-period model

In the general case, the discounted prices $\overline{S} = (\overline{S}^1, ..., \overline{S}^d)$ is given by

$$\overline{S}_n = \prod_{i=0}^{n} (1 + r_i)^{-1} S_n, \quad n \geq -1,$$

and the corresponding discounted value process of a strategy π defined by

$$\overline{V}_n^\pi = \prod_{i=0}^{n} (1 + r_i)^{-1} V_n^\pi, \quad n \geq -1.$$

The arbitrage-freeness is equivalent to the existence of an equivalent martingale measure $\mathbb{Q}$ such that the probabilities $q_n^i = \mathbb{Q}(X_n = i), \ n \in \mathbb{N}$ solve the linear equations

$$\begin{cases} q_{n+1}^0 M_{n+1}^0 S_n + ... + q_{n+1}^l M_{n+1}^l S_n = r_{n+1} S_n \\ q_n^0 + ... + q_n^l = 1. \end{cases}$$

The arbitrage-free market model is complete if and only if $l = d$ and the matrix

$$\begin{pmatrix} M^0_{n+1} S_n & \cdots & M^d_{n+1} S_n \\ 1 & \cdots & 1 \end{pmatrix} \in \mathcal{M}_{d+1}(\mathbb{R}),$$

is invertible.

Example 2.11.9. *Consider*

$$M^i_k = diag(\lambda^{1,i}_k, \ldots, \lambda^{d,i}_k) \in \mathcal{M}_d(\mathbb{R}), \quad \textit{for } k \in \{0, ..., N\} \textit{ and } i \in \{0, ..., d\}$$

such that $\lambda^{j,i}_k > -1$ *and the matrix*

$$A_k = \begin{pmatrix} \lambda^{1,0}_k & \cdots & \lambda^{1,d}_k \\ \vdots & & \vdots \\ \lambda^{d,0}_k & \cdots & \lambda^{d,d}_k \\ 1 & \cdots & 1 \end{pmatrix} \in \mathcal{M}_{d+1}(\mathbb{R}),$$

is invertible.

Our goal is to provide a solution to the hedging problem : each $\mathcal{F}$-measurable random variable F can be hedged by a self-financing strategy. In other terms, there is a self-financing π such that

$$V^\pi_N = F \quad a.s.$$

In order to simplify the exposition, without losing much in generality, we assume that all r_i are equal to r.

Proposition 2.11.10. *The portfolio π is self-financing if and only if,*

$$V^\pi_n = V^\pi_{-1} + \sum_{k=0}^{n} \beta_k \Delta B_{k-1} + < \gamma_k, \Delta S_{k-1} > . \tag{2.11.4}$$

Proof. If π is self-financing portfolio, it suffices to write

$$V^\pi_n = V^\pi_{-1} + \sum_{k=0}^{n} V^\pi_k - V^\pi_{k-1}.$$

Conversely, from (2.11.2) we have

$$\Delta V^\pi_n = \beta_{n+1} B_n + < \gamma_{n+1}, S_n > - \beta_n B_{n-1} - < \gamma_n, S_{n-1} > .$$

But, the relation (2.11.4) implies

$$\Delta V^\pi_n = \beta_n \Delta B_{n-1} + < \gamma_n, \Delta S_{n-1} >$$

hence,

$$B_n \Delta \beta_n + < S_n, \Delta \gamma_n > = 0.$$

□

Proposition 2.11.11. *If the portfolio π is self-financing, then*

$$\Delta\overline{V}_n^\pi =< \gamma_{n+1}, \Delta\overline{S}_n >, \quad n \geq -1. \tag{2.11.5}$$

Proof. We write

$$\begin{aligned} \Delta\overline{V}_n^\pi &= (1+r)^{-n-1}V_n^\pi - (1+r)^{-n}V_{n-1}^\pi \\ &= (1+r)^{-n-1} < \gamma_{n+1}, S_n > -(1+r)^{-n} < \gamma_{n+1}, S_{n-1} > . \end{aligned}$$

□

The identity 2.11.5 implies that

Corollary 2.11.12.

$$\overline{V}_n^\pi - \overline{V}_{n-1}^\pi = (1+r)^{-n-1}\sum_{j=1}^{d}(S_n^j - (1+r)S_{n-1}^j)\gamma_n^j.$$

Proposition 2.11.13. *Assume that the portfolio π is self-financing. Then we have the decomposition*

$$V_n^\pi = (1+r)^{n+1}V_{-1}^\pi + \sum_{k=0}^{n}\sum_{j=1}^{d}(1+r)^{n-k}(\lambda_k^{j,X_k} - r_k)\gamma_k^j S_{k-1}^j.$$

Proof. We write

$$\begin{aligned} V_n^\pi - V_{n-1}^\pi &= \beta_n(B_n - B_{n-1}) + < \gamma_n, S_n - S_{n-1} > \\ &= r\beta_n B_{n-1} + \sum_{j=1}^{d} \lambda_n^{j,X_n}\gamma_n^j S_{n-1}^j \\ &= rV_{n-1}^\pi + \sum_{j=1}^{d} \lambda_n^{j,X_n}\gamma_n^j S_{n-1}^j. \end{aligned}$$

□

The following proposition gives the unique equivalent martingale measure such that the market model is complete.

Proposition 2.11.14. *The process $(\overline{S}_n)_n$ is a d-dimensional martingale with respect to $(\mathcal{F}_n)_{-1\leq n\leq N}$ under the probability $\mathbb{Q}$ given by*

$$\begin{pmatrix} q_k^0 \\ \vdots \\ q_k^d \end{pmatrix} = A_k^{-1}\begin{pmatrix} r \\ \vdots \\ r \\ 1 \end{pmatrix}, \quad k \in \mathbb{N}.$$

Equivalently

$$\mathbb{E}_{\mathbb{Q}}[S_{n+1} \,|\mathcal{F}_n] = (1+r)S_n, \qquad n \geq -1,$$

where $\mathbb{E}_{\mathbb{Q}}$ denotes the expectation under $\mathbb{Q}$.

Recall that under this probability measure there is arbitrage freeness and the market is complete. By the predictable representation property for discrete-time martingales we have

$$\overline{S}^i_n = \overline{S}^i_{-1} + \sum_{k=0}^{n} < \mathbb{E}_{\mathbb{Q}}[D_k \overline{S}^i_k \,|\mathcal{F}_{k-1}], Y_k > .$$

So if π is a self-financing portfolio, (2.11.5) yields

$$\Delta \overline{V}^\pi_n \;=\; \sum_{i=1}^{d} \gamma^i_{n+1} < \mathbb{E}_{\mathbb{Q}}[D_n \overline{S}^i_n \,|\mathcal{F}_{n-1}], Y_n > . \tag{2.11.6}$$

The next result provides a self-financing strategy π as solution to the hedging problem.

Proposition 2.11.15. *Let $F \in \mathcal{L}^2(\Omega, \mathcal{F})$. For $n \in \{0, ..., N\}$, consider*

$$\gamma^i_n = (1+r)^{n-N} \frac{1}{(S^i_n - (1+r)S^i_{n-1})} \mathbb{E}_{\mathbb{Q}}[D^i_n F \,|\mathcal{F}_{n-1}], \quad i \in \{1, ..., d\}, \tag{2.11.7}$$

and

$$\beta_n = (1+r)^{-N-1}\mathbb{E}_{\mathbb{Q}}[F \,|\mathcal{F}_n] - (1+r)^{-n-1} < \gamma_n, S_n > . \tag{2.11.8}$$

Then the portfolio $\pi = (\beta, \gamma)$ is self-financing and satisfies

$$V^\pi_n = (1+r)^{n-N}\mathbb{E}_{\mathbb{Q}}[F \,|\mathcal{F}_n], \quad 0 \leq n \leq N,$$

in particular $V^\pi_N = F$ a.s, hence π is a hedging strategy leading to F.

Proof. Let $(\gamma_n)_{-1\leq n\leq N}$ be defined by (2.11.7) with $\gamma_{-1} = 0$, and consider the process $(\beta_n)_{-1\leq n \leq N}$ defined by

$$\begin{cases} \beta_{k+1} = \beta_k - (1+r)^{-k-1} < \Delta\gamma_k, S_k >, \quad k = -1, ..., N-1 \\ \beta_{-1} = (1+r)^{-N-1}\mathbb{E}_{\mathbb{Q}}[F] \end{cases},$$

such that $\pi = (\beta_n, \gamma_n)_{-1\leq n\leq N}$ satisfies the self-financing condition. Let

$$\overline{V}^\pi_{-1} = (1+r)^{-N-1}\mathbb{E}_{\mathbb{Q}}[F],$$

hence, by the corollary (9.1),

$$\overline{V}^\pi_n - \overline{V}^\pi_{n-1} = (1+r)^{-n-1} \sum_{j=1}^{d} (S^j_n - (1+r)S^j_{n-1})\gamma^j_n.$$

On the other hand, from the Clark-Ocone formula, we have

$$\begin{aligned}
&(1+r)^{-N-1}\mathbb{E}_{\mathbb{Q}}[F\,|\mathcal{F}_n] \\
&= (1+r)^{-N-1}\mathbb{E}_{\mathbb{Q}}[F] + \sum_{k=0}^{n}\sum_{j=1}^{d}(1+r)^{-N-1}\mathbb{E}_{\mathbb{Q}}[D_k^j F\,|\mathcal{F}_{k-1}]Y_k^j \\
&= (1+r)^{-N-1}\mathbb{E}_{\mathbb{Q}}[F] + \sum_{k=0}^{n}\sum_{j=1}^{d}(1+r)^{-k-1}(S_n^j - (1+r)S_{n-1}^j)\gamma_n^j.
\end{aligned}$$

Hence

$$V_n^\pi = (1+r)^{n-N}\mathbb{E}_{\mathbb{Q}}[F\,|\mathcal{F}_n].$$

In particular we have $V_N^\pi = F$. Note that the relation (2.11.8) follows from

$$V_n^\pi = \beta_n B_n + < \gamma_n, S_n >, \quad 0 \le n \le N.$$

$\square$

Chapitre 3

On the spectral distribution of the free Jacobi process

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Abstract : In this paper, we describe the spectral distribution of the free Jacobi process associated with the parameter values $\lambda = 1$, $\theta = 1/2$ and starting at the unit of the compressed probability space where it takes values. To proceed, we derive a time-dependent recurrence equation for its moments (actually valid for all parameter values) or equivalently a nonlinear partial differential equation (PDE) for its moment generating function. Then, we solve this PDE and expand the obtained solution around the origin. Doing so leads to an explicit formula for the moments, which shows that the free Jcobi process is ditributed at any time t as $(1/4)(2 + Y_{2t} + Y_{2t}^\star)$, where Y is a free unitary Brownian motion. We recover this formula relying on enumeration techniques toghether with the folowing result : if a is a symmetric Bernoulli random variable which is free from $\{Y, Y^\star\}$, then the distributions of Y_{2t} and that of $aY_taY_t^\star$ coincide. We close the exposition by investigating the spectral distribution of the free Jacobi process associated with the parameter sets : $\lambda = 1, \theta \in (0, 1]$ and $\lambda \in (0, 1], \theta = 1/2$.

3.1 Reminder and motivation

The one dimensional Jacobi process is a two parameters-dependent stationary random motion valued in the interval $[0, 1]$, and its equilibrium measure is given by the Beta distribution ([11]). For special values of the parameters, this process can be realized as norms of projections of a spherical Brownian motion in a finite-dimensional Euclidean space onto spheres of lower dimensions ([3]). When extended to matrix spaces, this realization gives rise to matrix Jacobi processes as radial parts of corners of Brownian motions in the orthogonal and the unitary groups, and the corresponding equilibrium measure is the matrix-variate Beta distribution ([9], [14]). With free probability theory as motivation, the free Jacobi process was realized in [12] as the rescaled limit (in the sense of noncommutative moments) of the complex matrix Jacobi process. Its equilibrium distribution was determined in [8], [9] and [12]. By the virtue of asymptotic freeness of independent unitarily-invariant random matrices and constant ones whose distributions converge, the matrix model of the free Jacobi process suggests the following abstract definition (see [12] for details). Consider a noncommutative W*-probability space $(\mathscr{A}, \tau)$, that is, a von Neumann algebra $\mathscr{A}$ with unit $\mathbf{1}$ and endowed with a faithful normalized trace τ. Let $\theta \in (0, 1), \lambda > 0$ such that $0 < \lambda\theta < 1$ and take two projections P, Q in $\mathscr{A}$ such that

- $\tau(P) = \lambda\theta, \tau(Q) = \theta$,
- $PQ = QP = P$ $(P \leq Q)$ if $\lambda \leq 1$ and $PQ = QP = Q$ $(P \geq Q)$ otherwise.

Let also Y be a free unitary Brownian motion in $\mathscr{A}$ ([5]), and assume

$$\{P, Q\},\ \{Y, Y^\star\}$$

are free families in $\mathscr{A}$. Then

$$J_t := PY_tQY_t^\star P$$

defines the free Jacobi process of parameters (λ, θ), and one easily sees from

$$P - J_t = PY_t(\mathbf{1} - Q)Y_t^\star P$$

that $P - J$ is a free Jacobi process too, of parameters $(\lambda\theta/(1 - \theta), 1 - \theta)$. However, the stochastic analysis of J performed in [12] requires that J_0 and $P - J_0$ are injective operators in the compressed probability space

$$(P\mathscr{A}P, \frac{1}{\tau(P)}\tau).$$

For that reason, the free Jacobi process studied in [12] and denoted J as well is driven by a free unitary Brownian motion starting at Z, where Z is a unitary operator in $\mathscr{A}$ such that

$$\{P, Q\},\ \{Z, Z^\star\}, \{Y, Y^\star\}$$

are free families in $\mathscr{A}$. Actually, the operator Z has to ensure that

$$T := \inf\{t > 0,\ J_t \,\text{or}\, P - J_t \text{ is non injective}\}$$

is positive and the issue of our stochastic analysis is the free stochastic differential equation

$$dJ_t = \sqrt{\lambda\theta}\sqrt{J_t}dW_t\sqrt{P - J_t} + \sqrt{\lambda\theta}\sqrt{J_t}dW_t^{\star}\sqrt{P - J_t} + (\theta P - J_t)dt, \quad 0 \leq t < T, \tag{3.1.1}$$

where W is a $P\mathscr{A}P$-complex free Brownian motion. This equation was the key ingredient for investigating spectral properties of J_t. More precisely, let $m_n(t) := \tau(J_t^n)/\tau(P)$ be the moments of J_t in $P\mathscr{A}P$; then the following holds :

$$\partial_t m_n(t) = -nm_n(t) + \theta n m_{n-1}(t) + \lambda\theta n \sum_{k=0}^{n-2} m_{n-k-1}(t)(m_k(t) - m_{k+1}(t)) \tag{3.1.2}$$

$$m_0(t) = 1, m_n(0) = \tau((PZQZ^{\star})^n)/\tau(P),$$

for any $n \geq 1$ and up to time T (the sum in the RHS of (3.1.2) is taken to be empty when $n = 1$). On the one hand, we observe that the infinite-dimensional system (3.1.2) is described by an explicit scheme, and consequently the moments $(m_n(t))_{n\geq 0}$ should be well defined globally in time. This is in contrast with the free stochastic differential equation (3.1.1), which may develop finite-time blow-up solutions. As a matter of fact, (3.1.2) may be interpreted as a 'weak' continuation of (3.1.1) beyond the life span T (therefore independently of Z). On the other hand, Theorem 3.4 in [4] supports the the validity of of this continuation. Indeed, this theorem supplies a the time-dependent recurrence equation for

$$\tau(a_1 Y_t a_2 Y_t^{\star} \dots a_{2n-1} Y_t a_{2n} Y_t^{\star}),$$

where $\{a_1, \dots, a_{2n}\}$ is a $2n$-tuple of random variables forming a $\star$-free family with Y. Accordingly, we specialized to this theorem to $a_k = Z^{\star}PZ$ when k is odd and $a_k = Q$ when k is even, it should lead to (3.1.2) since both families $\{Z^{\star}PZ, Q\}$ and $\{Y, Y^{\star}\}$ are free. It also suggests that (3.1.2) still holds, yet with a different initial value $m_n(0)$, if we rather consider $a_k = P$ when k is odd since $Z^{\star}PZ$ and P have the same moments. Based on these ideas, we shall work out the time-dependent recurrence equation stated in Theorem 3.4 in [4] and actually prove that holds for any unitary Z that is $\star$-free from Y. Once we do so, we proceed to the study of the spectral distribution of $J = PYQY^{\star}P, Y_0 = \mathbf{1}$, which was discarded in [12]. To this end, we notice that for any λ, the sequence $v_n(t) := \lambda m_n(t), n \geq 1$ satisfies

$$\partial_t v_n(t) = -nv_n(t) + \theta n v_{n-1}(t) + \theta n \sum_{k=0}^{n-2} v_{n-k-1}(t)(v_k(t) - v_{k+1}(t))$$

with initial data $v_n(0) = \lambda m_n(0), v_0(t) = \lambda m_0(t)$. Consequently, we shall primarily consider the value $\lambda = 1$ and focus on the corresponding free Jacobi process. However, due to the high nonlinearity of (3.1.2), we further restrict our attention to the value $\theta = 1/2$. Note in passing that the matrix model corresponding to these parameter value is the radial part of square upper-left corners of large unitary Brownian motions ($\lambda = 1$) and that the sizes of these corners are asymptotically halfs of those of the unitary Brownian motions ($\theta = \frac{1}{2}$). Then, our main result states that the spectral distribution of J_t in the compressed space, say μ_t, fits the distribution of the random variable

$$\frac{1}{4}[Y_{2t} + Y_{2t}^\star + 2\mathbf{1}]$$

in $(\mathscr{A}, \tau)$. In particular μ_t is absolutely continuous with respect to Lebesgue measure on the real line and its support fill in the interval $(0, 1)$ at time $t = 2$. Actually, our description follows from a closed formula for $m_n(t), n \geq 1$, namely :

$$m_n(t) = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{k=1}^{n}\binom{2n}{n-k}\frac{1}{k}L_{k-1}^1(2kt)e^{-kt} \tag{3.1.3}$$

where L_n^1 is the n-th Laguerre polynomial of index 1 ([37]). Formula (3.1.3) is in turn obtained after solving a nonlinear partial differential equation (hereafter p.d.e) for the moment generating function of μ_t :

$$M_t(z) = \sum_{n\geq 0} m_n(t)z^n \quad |z| < 1.$$

In fact, by the virtue of (3.1.2), this p.d.e admits a unique solution in the class of analytic functions around zero. Nevertheless, the binomial coefficients present in (3.1.3) suggests that this formula might be derived based on enumeration techniques. To this aim, we start by noting that when $\lambda = 1$, the faithfulness of τ and $P \leq Q$ entail $P = Q$: indeed $\tau(P) = \tau(Q)$ so that

$$\tau((P-Q)^2) = 2\tau(P) - 2\tau(PQ) = 0.$$

Next, if further $\theta = 1/2$ then $P = (1+a)/2$ where $a \in \mathscr{A}$ is a self-adjoint symmetric Bernoulli random variable and we shall derive the following expansion :

$$2\tau((\mathbf{1}+a)Y_t(\mathbf{1}+a)Y_t^\star)^n) = \binom{2n}{n} + \sum_{k=1}^{n}\binom{2n}{n-k}\tau((aY_taY_t^\star)^k).$$

Finally, we recover (3.1.3) after proving that Y_{2t} and $v_t := aY_taY_t^\star$ have the same distribution. At this level, the following comments on this last result should be pointed out in relation to Theorem 3.4 in [4] : Both proofs are similar and rely on the use of p.d.e satisfied by bthe moment generating function of Y_t. Besides, both variable $aY_{2t}aY_{2t}^\star$ and $(aY_tbY_t^\star)^2$ have the same distribution. Here $b \in \mathcal{A}$ is a self-adjoint symmetric Bernoulli random variable and is independant of a. This independence is necessary for obtaining the result stated in Theorem 3.6 in [4] in the same way the fact that $a^2 = 1$ is for obtaining ours. We close the paper by investigating the spectral distribution of J_t associated with the parameter values $\theta \in (0,1)$ and $\lambda = 1$. Indeed, th enumeration techniques used to derive (3.1) not only provide another elegant proof of the description of μ_t, but also open the way to investigate the spectral distribution of J_t for arbitrary $\theta \in (0,1)$ and $\lambda = 1$. More precisely, the expansion (refexpan) comes in with the additional term $2^{2n-1}(2\theta - 1)$. However, the spectral distribution of $aY_taY_t^\star$ is unknown to the best of our knowledge. The paper is organized as follows. In section 2, we prove that (3.1.2) holds for any unitary Z that is $\star$-free from Y. Computations are only given for $a_k = P$ when k is odd and $a_k = Q$ when k is even. Section 3 is devoted to the description of the spectral distribution μ_t of J_t in the particular case $\lambda = 1, \theta = 1/2$. In section 4, we prove (3.1) and recover this description. The last section exhibits some developments concerned with the spectral distribution of J_t associated with the parameter sets : $\lambda = 1, \theta \in (0,1]$ and $\lambda \in (0,1], \theta = 1/2$.

3.2 A recurrence time-dependent equation

In this section, we prove that

Proposition 3.2.1. *The moments of the free Jacobi process*

$$m_n(t) := \frac{1}{\tau(P)}\tau(PZY_tQY_t^\star Z^\star P)^n),\ n \geq 1$$

satisfy (3.1.2) *for any unitary Z that is $\star$-free from Y and from $\{P, Q\}$.*

Proof : As explained in the introductory part, computations are given in the special case $Z = \mathbf{1}$. Moreover, we shall equivalently prove that the sequence

$$r_n(t) := \tau(J_t^n) = \tau(P)m_n(t), n \geq 1$$

satisfies

$$\partial_t r_n(t) = -nr_n(t) + n\theta r_{n-1}(t) + n\sum_{k=0}^{n-2} r_{n-k-1}(t)(r_k(t) - r_{k+1}(t)) \tag{3.2.4}$$

with $r_0(t) = \tau(P) = \lambda\theta$. To proceed, we recall Theorem 3.4 in [4] :

Theorem 3.2.2. *Let $n \geq 1$ and define*

$$f_{2n}(a_1, \ldots, a_{2n}, t) := e^{nt}\tau(a_1 Y_t a_2 Y_t^\star \ldots a_{2n-1} Y_t a_{2n} Y_t^\star)$$

where $\{a_1, \ldots, a_{2n}\} \in \mathscr{A}$ is $\star$-free with Y. Set $f_0(A, t) := \tau(A)$ for any $A \in \mathscr{A}$ then

$$\partial_t f_{2n}(a_1, \ldots, a_{2n}, t) = -\sum_{\substack{1 \leq k < l \leq 2n \\ l-k \equiv 0[2]}} f_{2n-(l-k)}(a_1, \ldots, a_k, a_{l+1}, \ldots, a_{2n}, t) f_{l-k}(a_{k+1}, \ldots, a_l, t) +$$

$$e^t \sum_{\substack{1 \leq k < l \leq 2n \\ l-k-1 \equiv 0[2]}} f_{2n-(l-k)-1}(a_1, \ldots, a_{k-1}, a_k a_{l+1}, a_{l+2}, \ldots, a_{2n}, t) f_{l-k-1}(a_l a_{k+1}, a_{k+2}, \ldots, a_{l-1}, t)$$

Now, we specialize Theorem 3.2.2 to $a_k = P$ if k is odd and $a_k = Q$ otherwise. Then the result is straightforward when $n = 1$ and is even stated in [4], p.923. So let $n \geq 2$ and note that both indices k, l in the first (respectively second) sum in Theorem 3.2.2 have the same (respectively different) parity, therefore k and $l+1$ in the second sum have the same parity and so do l and $k+1$. Accordingly, the first sum does not contain terms $f_0(\cdot, t)$ while the second does : they correspond to indices $l = 2n, k = 1$ and to $l = k+1, 1 \leq k \leq 2n-1$. Since P and Q are idempotent and since τ is a trace, then the contribution of indices $k = 1, l = 2n$ is $\tau(P) f_{2n-2}(P, Q, \ldots, P, Q)$ while that of $l = k+1, 1 \leq k \leq 2n-1$ is

$$[n\tau(Q) + (n-1)\tau(P)] f_{2n-2}(P, Q, \ldots, P, Q, t)$$

respectively. Thus, both contributions sum up to

$$n(\tau(Q) + \tau(P)) f_{2n-2}(P, Q, \ldots, P, Q, t). \tag{3.2.5}$$

Next we write $l = k + 2s + 1$ for integer positive values of s and distinguish $n = 2$ and $n \geq 3$. If $n = 2$ then there is no additional term in the second sum, while if $n \geq 3$ we separate $k = 1$ and $2 \leq k \leq 2n-3$. By the same properties of P, Q, τ mentioned above, the contribution of indices $k = 1, l = 2s + 2$ is

$$\sum_{s=1}^{n-2} f_{2(n-s-1)}(P, Q, \ldots, P, Q, t) f_{2s}(P, Q, \ldots, P, Q, t). \tag{3.2.6}$$

For the remaining values of $2 \leq k \leq 2n-3$, we distinguish even and odd ones : the contribution of indices $k = 2j, 1 \leq j \leq n-2, l = 2j + 2s + 1$ is

$$\sum_{j=1}^{n-2} \sum_{s=1}^{n-j-1} f_{2(n-s-1)}(P, Q, \ldots, P, Q, t) f_{2s}(P, Q, \ldots, P, Q, t)$$

while that of $k = 2j+1, 1 \leq j \leq n-2, l = 2s + 2j + 2$ is also

$$\sum_{j=1}^{n-2} \sum_{s=1}^{n-j-1} f_{2(n-s-1)}(P, Q, \ldots, P, Q, t) f_{2s}(P, Q, \ldots, P, Q, t).$$

By rearranging the terms in both obtained double sums, we see that the contribution of indices $2 \leq k \leq 2n-3, l = 2k+2s+1$ is

$$2\sum_{s=1}^{n-2}(n-s-1)f_{2(n-s-1)}(P,Q,\dots,P,Q,t)f_{2s}(P,Q,\dots,P,Q,t)$$

which simplify after the index change $s \mapsto n-s-1$ to

$$(n-1)\sum_{s=1}^{n-2} f_{2(n-s-1)}(P,Q,\dots,P,Q,t)f_{2s}(P,Q,\dots,P,Q,t). \tag{3.2.7}$$

Similarly we consider the first sum in Theorem 3.2.2 and write $l = k+2s, 1 \leq k \leq 2n-2, 1 \leq s \leq n-[(k+1)/2], n \geq 2$. Then it contributes to

$$n\sum_{s=1}^{n-1} f_{2(n-s)}(P,Q,\dots,P,Q,t)f_{2s}(P,Q,\dots,P,Q,t). \tag{3.2.8}$$

Now we recall that $f_{2n}(P,Q,\dots,P,Q,t) = e^{nt}r_n(t)$ and take into account the exponential factor in front of the second sum of Theorem 3.2.2. If $n \geq 3$ then (3.2.7) and (3.2.8) sum up to

$$e^{nt}[-nr_{n-1}(t)r_1(t) - \sum_{s=1}^{n-2} r_{n-s-1}(t)r_s(t) + n\sum_{s=1}^{n-2} r_s(t)[r_{n-s-1}(t) - r_{n-s}(t)].$$

With regard to (3.2.5), (3.2.6), the whole contribution of the RHS of Theorem 3.2.2 is

$$e^{nt}\{n\tau(Q) + n[\tau(P) - r_0(t)]r_{n-1}(t) + n\sum_{s=1}^{n-1} r_s(t)[r_{n-s-1}(t) - r_{n-s}(t)]\}.$$

But $\partial_t f_{2n}(t) = e^{nt}[\partial_t r_n t + nr_n(t)]$ together with $r_0(t) = \tau(P)$ show that (3.2.4) holds for any $t > 0$ and any $n \geq 3$. If $n = 2$ then the whole contribution is given by (3.2.5) and (3.2.8) leading to (3.2.4) as well. ∎

3.3 The case $\theta = 1/2, \lambda = 1$

This section is devoted to the description of the spectral distribution μ_t of J_t when $\lambda = 1, \theta = 1/2$ and $J_0 = P$. A major step towards it is the following result :

Proposition 3.3.1. *Let L_k^1 be the k-th Laguerre polynomial and let*

$$\rho_t(z) := \sum_{k=1}^{\infty} \frac{1}{k} L_{k-1}^1(kt)z^k,\ |z| < 1,$$

be the unique solution of ([5], [36])

$$\partial_t \rho_t + \frac{z}{2}\partial_z \rho_t^2 = 0, \quad \rho_0(z) = \frac{z}{1-z}$$

in the class of analytic function around zero. Then the moment generating function M_t of μ_t is given by

$$M_t(z) = \frac{1}{\sqrt{1-z}}\left\{1 + 2\rho_{2t}\left(\frac{ze^{-t}}{(\sqrt{1-z}+1)^2}\right)\right\}, \, |z| < 1.$$

Proof : Before coming through computations, we point out that

$$\alpha : z \mapsto \frac{z}{(1+\sqrt{1-z})^2}$$

maps the open unit disc into itself. Indeed, the following expansion holds ([37] p.70)

$$\frac{1}{(1+\sqrt{1-z})^2} = \frac{1}{4}\sum_{k\geq 0}\frac{(1)_n(3/2)_n}{(3)_n n!}z^n, \, |z| < 1$$

and is still convergent for $z = 1$ by Gauss Theorem ([37], p.49). Hence the expression of $M_t(z)$ given in the proposition makes sense for all $|z| < 1$. Now, let $|z| > 1$ then similar computations leading to Proposition 7.1. in [12] shows that

$$G_t(z) := \frac{1}{z}\sum_{n=0}^{\infty}\frac{m_n(t)}{z^n},$$

satisfies the p.d.e. for all (λ, θ)

$$\partial_t G_t = \partial_z \left\{[(1-2\lambda\theta)z - \theta(1-\lambda)]G_t + \lambda\theta z(z-1)G_t^2\right\}$$

with initial value $G_0(z) = 1/(z-1)$. This p.d.e. simplifies when one substitutes $\lambda = 1, \theta = 1/2$ to

$$\partial_t G_t = \frac{1}{2}\partial_z \left\{z(z-1)G_t^2\right\}, \, G_0(z) = \frac{1}{z-1}$$

or equivalently for $|z| < 1$

$$\partial_t M_t = -\frac{z}{2}\partial_z \left\{(1-z)M_t^2\right\}, \, M_0(z) = \frac{1}{1-z}.$$

Set $S_t(z) := \sqrt{1-z}M_t(z) - 1$, then

$$\partial_t S_t + z\sqrt{1-z}\partial_z S_t + \frac{1}{2}z\sqrt{1-z}\partial_z S_t^2 = 0$$

with the initial data

$$S_0(z) = \frac{1}{\sqrt{1-z}} - 1.$$

Next note that α is invertible in a neighborhood of zero with inverse function given by

$$\alpha^{-1}(z) = \frac{4z}{(1+z)^2}.$$

Keeping in mind that $|\alpha(z)| \leq |z| < 1$ in the open unit disc, then α extends to a biholomorphic map from the open unit disc onto its image. Moreover,

$$z\sqrt{1-z}\alpha'(z) = \alpha(z).$$

Hence $F_t : z \mapsto S_t(\alpha^{-1}(z))$ and $w_t : z \mapsto F_t(e^t z)$ satisfy

$$\partial_t F_t + z\partial_z F_t + \frac{1}{2}z\partial_z F_t^2 = 0$$

and

$$\partial_t w_t + \frac{1}{2}z\partial_z w_t^2 = 0.$$

Finally one easily checks from

$$1+\alpha(z) = \frac{2}{1+\sqrt{1-z}}, \quad 1-\alpha(z) = \frac{2\sqrt{1-z}}{1+\sqrt{1-z}}$$

that

$$w_0(\alpha(z)) = S_0(z) = \frac{2\alpha(z)}{1-\alpha(z)} = 2\rho_0(\alpha(z)) \Leftrightarrow w_0(z) = 2\rho_0(z).$$

Since $2\rho_{2t}$ and w_t satisfy the same partial differential equation as ρ_t, the proposition is proved. ■

The moments $m_n(t)$ are given by

Corollary 3.3.2. *For any $n \geq 1$ and any $t \geq 0$*

$$m_n(t) = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{1}{2^{2n-1}}\sum_{k=1}^{n}\binom{2n}{n-k}\frac{1}{k}L_{k-1}^{1}(2kt)e^{-kt}.$$

Proof : First of all the generalized binomial Theorem yields

$$\frac{1}{\sqrt{1-z}} = \sum_{n=0}^{\infty}\frac{(1/2)_n}{n!}z^n, \ |z| < 1$$

where $(1/2)_n = \Gamma(n+1/2)/\Gamma(1/2)$ is the Pochhammer symbol. But Legendre duplication formula ([18]) shows that

$$\frac{(1/2)_n}{n!} = \frac{1}{2^{2n}}\binom{2n}{n}.$$

Now let ${}_2F_1$ be the Gauss hypergeometric function then for any $k \geq 1$ ([37] p.70)

$$\begin{aligned}\frac{2^{2k}}{\sqrt{1-z}}\frac{1}{(\sqrt{1-z}+1)^{2k}} &= {}_2F_1\left(k+\tfrac{1}{2}, k+1, 2k+1, z\right) \\ &= \textstyle\sum_{n=0}^{\infty} \frac{(k+1/2)_n (k+1)_n}{(2k+1)_n}\frac{z^n}{n!}\end{aligned}$$

Using Legendre duplication formula again, we derive

$$(2n+2k)! = \frac{2^{2n+2k}}{\sqrt{\pi}}\Gamma(n+k+1/2)\Gamma(n+k+1)$$

which yields

$$\begin{aligned}\sum_{n=0}^{\infty}\binom{2n+2k}{n}\frac{z^n}{2^{2n}} &= \tfrac{2^{2k}\Gamma(k+1/2)\Gamma(k+1)}{\sqrt{\pi}\Gamma(2k+1)}\textstyle\sum_{n=0}^{\infty}\frac{(k+1/2)_n(k+1)_n}{(2k+1)_n}\frac{z^n}{n!} \\ &= \tfrac{2^{2k}}{\sqrt{1-z}}\tfrac{1}{(\sqrt{1-z}+1)^{2k}}.\end{aligned}$$

As a result

$$\begin{aligned}\frac{1}{\sqrt{1-z}}\rho_{2t}\left(\frac{ze^{-t}}{(\sqrt{1-z}+1)^2}\right) &= \frac{1}{\sqrt{1-z}}\sum_{k=1}^{\infty}\frac{1}{k}L_{k-1}^1(2kt)\left[\frac{ze^{-t}}{(\sqrt{1-z}+1)^2}\right]^k \\ &= \sum_{k=1}^{\infty}\frac{1}{k}L_{k-1}^1(2kt)e^{-kt}\sum_{n=0}^{\infty}\binom{2n+2k}{n}\frac{z^{n+k}}{2^{2n+2k}} \\ &= \sum_{k=1}^{\infty}\frac{1}{k}L_{k-1}^1(2kt)e^{-kt}\sum_{n=k}^{\infty}\binom{2n}{n-k}\frac{z^n}{2^{2n}}.\end{aligned}$$

The Corollary is proved. ■

We are now ready to give the description of μ_t :

Corollary 3.3.3. *When $\theta = 1/2, \lambda = 1$, the free Jacobi process starting at P is distributed in the compressed probability space $(P\mathscr{A}P, 2\tau)$ as the random variable*

$$\frac{Y_{2t}^{-1}}{4}(1+Y_{2t})^2 = \frac{1}{4}[Y_{2t}^{-1}+2+Y_{2t}]$$

in $\mathscr{A}$. Consequently, μ_t is absolutely continuous with respect to Lebesgue measure on the real line and its density is given by

$$2\frac{k_{2t}(e^{2i\arccos(\sqrt{x})})}{\sqrt{x(1-x)}}\mathbf{1}_{[0,1]}(x)dx.$$

where k_t is the density of the spectral distribution of the free unitary Brownian motion. In particular, the support of μ_t is the whole interval $[0,1]$ for any time $t \geq 2$.

Proof : Let $h_n(t) = \tau(Y_t^n), n \geq 1$ be the moments of Y_t in $(\mathscr{A}, \tau)$, then ([5])

$$h_n(t) = \frac{e^{-nt/2}}{n} L_{n-1}^1(nt),$$

But since Y and $Y^\star = Y^{-1}$ have the same distribution then $h_n = h_{-n}, n \in \mathbb{Z}$. As a result

$$\begin{aligned}
m_n(t) &= \frac{\binom{2n}{n}}{2^{2n}} + \frac{2}{2^{2n}} \sum_{k=1}^{n} \binom{2n}{n-k} h_k(2t) \\
&= \frac{1}{2^{2n}} \left\{ \binom{2n}{n} + \sum_{k=1}^{n} \binom{2n}{n-k} h_k(2t) + \sum_{k=1}^{n} \binom{2n}{n+k} h_k(2t) \right\} \\
&= \frac{1}{2^{2n}} \left\{ \binom{2n}{n} + \sum_{k=1}^{n} \binom{2n}{n-k} h_k(2t) + \sum_{k=-n}^{-1} \binom{2n}{n-k} h_{-k}(2t) \right\} \\
&= \frac{1}{2^{2n}} \sum_{k=-n}^{n} \binom{2n}{n-k} h_k(2t) \\
&= \frac{1}{2^{2n}} \sum_{k=0}^{2n} \binom{2n}{2n-k} \tau(Y_{2t}^{k-n}) \\
&= \frac{1}{2^{2n}} \tau(Y_{2t}^{-n}(1 + Y_{2t})^{2n}).
\end{aligned}$$

Finally, the support of μ_t is entirely determined by the one of k_{2t} (see [7]).

Remark 3.3.4. *When* $\lambda = 1, \theta = 1/2$ (3.1.2) *takes the form*

$$\partial_t m_n(t) + \frac{n}{2} m_n(t) = \frac{n}{2} \sum_{k=0}^{n-1} m_{n-1-k}(t)[m_k(t) - m_{k+1}(t)].$$

Thus since the moments of Y_t *converge as* $t\infty$ *to those of a Haar unitary random variable, then the moments*

$$m_n(\infty) = \frac{1}{2^{2n}} \binom{2n}{n} := m_n$$

satisfy

$$m_n = \sum_{k=0}^{n-1} m_{n-k-1}[m_k - m_{k+1}]. \tag{3.3.9}$$

But if $C_k := (1/(k+1))\binom{2k}{k}$ *is the k-th Catalan number ([33]) then Legendre duplication formula entails*

$$m_k - m_{k+1} = \frac{1}{2\sqrt{\pi}}\frac{\Gamma(k+1/2)}{(k+1)!} = \frac{1}{2^{2k+1}}C_k.$$

As a result

$$\sum_{k=0}^{n-1} m_{n-k-1}[m_k - m_{k+1}] = \frac{2}{2^{2n}}\sum_{k=0}^{n-1}(n-k)C_{n-k-1}C_k = \frac{n+1}{2^{2n}}\sum_{k=0}^{n-1}C_{n-k-1}C_k.$$

Consequently, (3.3.9) *is nothing else but the recurrence relation for Catalan numbers :*

$$C_n = \sum_{k=0}^{n-1} C_{n-1-k}C_k.$$

3.4 Enumerative derivation of the moments

Recall that if $\lambda = 1$ then the faithfulness of τ forces $P = Q$. If further $\theta = 1/2$ then $P := (1+a)/2$ where $a = a^\star \in \mathscr{A}$ is distributed according to

$$\frac{1}{2}(\delta_1 + \delta_{-1}).$$

In the sequel, we shall derive (3.1.3) relying on enumeration techniques. More precisely,

Proposition 3.4.1. *Let a be a self-adjoint random variable in* $\mathscr{A}$ *distributed according to*

$$\frac{1}{2}(\delta_1 + \delta_{-1})$$

and assume a *and* Y *are* $\star$*-free. Then for any* $n \geq 1$

$$\tau[((1+a)Y_t(1+a)Y_t^\star)^n] = \frac{1}{2}\binom{2n}{n} + \sum_{k=1}^{n}\binom{2n}{n-k}\tau((aY_taY_t^\star)^k).$$

Proof : let $b := Y_taY_t^\star$ then

$$(1+a)Y_t(1+a)Y_t^\star = (1+a)(1+b)$$

therefore $[(1+a)(1+b)]^n$ consists of words formed by letters picked from the alphabet $\{a, b\}$, subject to the cancellations $\{a^{2k} = b^{2k} = \mathbf{1}, a^{2k+1} = a, b^{2k+1} = b\}$. Moreover, we claim that those formed by an odd number of letters have zero expectation. Indeed, any such word has a reduced (after taking into account cancellations and that there is no relation between a and b) expression either $aba\cdots aba$ or $bab\cdots bab$. The claim then follows from the trace property of τ and from $\tau(a)=\tau(b)=0$. As a result, we need to enumerate for each $n\geq 1$ words $\{(ab)^k, 1\leq k\leq n, (ba)^k, 1\leq k\leq n-1\}$ (see below for the constant term $k=0$). To this end, let $c(n,k), d(n,k), e(n,k)$ be the number of words $(ab)^k, (ab)^k a, (ba)^k$ respectively. Then the expansion

$$[(1+a)(1+b)]^n = [(1+a)(1+b)]^{n-1}(1+a+b+ab)$$

and the observation

$$(ab)^k = [(ab)^k]\mathbf{1} = [(ab)^k a]a = [(ab)^{k-1}a]b = [(ab)^{k-1}](ab)$$

give the recurrence relation

$$c(n,k) = c(n-1,k) + d(n-1,k) + d(n-1,k-1) + c(n-1,k-1),\ n\geq 2, k\geq 1$$

and

$$c(1,k) = \delta_{k0} + \delta_{k1}$$

where δ_{ij} is the Kronecker symbol. In a similar fashion,

$$(ab)^k a = [(ab)^k a]\mathbf{1} = [(ab)^k]a = [(ab)^{k+1}]b = [(ab)^{k+1}a](ab)$$

gives the recurrence relation

$$d(n,k) = d(n-1,k) + c(n-1,k) + c(n-1,k+1) + d(n-1,k+1),\ n\geq 2, k\geq 0,$$

with

$$d(1,k) = \delta_{k0},$$

while

$$[(1+a)(1+b)]^n = (1+a+b+ab)[(1+a)(1+b)]^{n-1}$$

together with

$$(ba)^k = \mathbf{1}(ba)^k = a[(ab)^k a] = b[(ab)^{k-1}a] = ab[(ba)^{k+1}]$$

give the recurrence relation

$$e(n,k) = e(n-1,k) + d(n-1,k) + d(n-1,k-1) + e(n-1,k+1),\ n\geq 3, k\geq 1,$$

with

$$e(2,k) = 3\delta_{k0} + \delta_{k1},\ e(1,k) = \delta_{k0}.$$

Now, we compute $c(n,0)$ which corresponds to the constant term (not depending on time t) in the expansion

$$\tau[((1+a)(1+b))^n].$$

To proceed, we argue that up to a factor, $c(n,0)$ is the n-th moment $m_n(\infty)$ of the stationary distribution of the free Jacobi process

$$2\frac{c(n,0)}{2^{2n}} = \frac{1}{2^{2n}}\binom{2n}{n} \Rightarrow c(n,0) = \frac{1}{2}\binom{2n}{n}.$$

Indeed, for fixed $n \geq 1$

$$\lim_{t\to\infty} \tau[(aY_taY_t^\star)^n] = \tau[(aUaU^\star)^n]$$

where U is a Haar unitary distributed random variable. But a and $UaU^\star$ are free in $\mathscr{A}$ and since $\tau(a) = \tau(UaU^\star) = 0$ then the very definition of freeness implies that $\tau[(aUaU^\star)^n] = 0$ for any $n \geq 1$. We can also compute $c(n,0)$ by considering the von Neumann algebra generated by $\{a,b\}$ endowed with the state that assigns the value 1 to the unit and vanishes otherwise. More precisely, this algebra is the free product of the von Neumann algebras generated by $\{a\}$ and by $\{b\}$ since the latters may be realized as von Neumann algebras of the cyclic group of order two so that Theorem 1.6.3. in [15] applies. Therefore a and b are free there and it is obvious that they are symmetric Bernoulli random variable with respect to the given state. It follows that $c(n,0)$ is the constant term in the n-th moment of the two-fold free multiplicative convolution

$$\frac{1}{2}(\delta_0+\delta_2) \boxtimes \frac{1}{2}(\delta_0+\delta_2)$$

which may be computed using the S-transform ([33]). Based on the initial values $c(n,0), d(1,0)$, we proceed by mutual induction (on k for fixed n then on n) and use the elementary identity

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

in order to prove that

$$\begin{aligned} c(n,k) &= \binom{2n-1}{n-k} \\ d(n,k) &= \binom{2n-1}{n-k-1} = e(n,k). \end{aligned}$$

Note by passing that $d(n,k-1) = c(n,k)$ which agrees with the recurrence relations for $d(n,k)$ and $e(n,k)$. Finally

$$c(n,k) + e(n,k) = \binom{2n-1}{n-k} + \binom{2n-1}{n-k-1} = \binom{2n}{n-k}. \blacksquare$$

In order to recover (3.1.3), it suffices to observe the following result.

Lemma 3.4.2. *The unitary random variable $aY_taY_t^\star$ is distributed as Y_{2t}.*

Proof : since the distribution of a unitary random variable is entirely determined by its moments, we shall prove that the sequence defined by

$$s_n(t) := e^{nt}\tau((aY_taY_t^\star)^n),\, n \geq 1,$$

is the (unique) solution of

$$\partial_t s_n(t) = -n\sum_{k=1}^{n-1} s_{n-k}(t)s_k(t), \quad n \geq 2,\, s_1(t) = 1,$$

then infer from [36] (Theorem 4 p. 669) that

$$s_n(t) = \frac{1}{n}L_{n-1}^1(2nt).$$

To proceed, we specialize Theorem 3.2.2 to $a_k = a$ for all k and notice that the second sum vanishes since $a^2 = \mathbf{1}$ and since $\tau(a) = 0$. The contribution of the first sum is easily seen to be

$$-n\sum_{k=1}^{n-1} s_{n-k}(t)s_k(t)$$

and the value of $s_1(t) = 1$ is readily derived from [4] p. 923 by using $\tau(a) = 0, \tau(a^2) = 1$. The lemma is proved. ■

3.5 Further developments : general parameter values

In this section, we investigate the spectral distribution of J_t for more general parameter sets : $\lambda = 1, \theta \in (0, 1]$ and $\lambda \in (0, 1], \theta = 1/2$.

3.5.1 $\lambda = 1, \theta \in (0, 1]$

As one easily realizes, the enumeration techniques used to expand

$$\tau[((1 + a)(1 + b))^n]$$

remain valid when $\lambda = 1, \theta \in (0, 1)$. Indeed, $P = (1 + a)/2$ where the spectral distribution of a is

$$\theta\delta_1 + (1 - \theta)\delta_{-1},$$

and we need to take into account the contribution of words with odd number of letters. By the trace property of τ and the relations $a^2 = b^2 = \mathbf{1}$, this contribution is $\tau(a) = (2\theta - 1)$ up to a positive integer say $c(n)$. Therefore

$$\tau[((1+a)Y_t(1+a)Y_t^\star)^n] = \frac{1}{2}\binom{2n}{n} + \sum_{k=1}^{n}\binom{2n}{n-k}\tau((aY_taY_t^\star)^k) + (2\theta-1)c(n).$$

The coefficient $c(n)$ is easily computed when $\theta \in (0,1) \setminus \{1/2\}$ by letting $t = 0$ and using $(1+a)^2 = 4P, a^2 = \mathbf{1}$:

$$4^n\theta = 2^{2n-1} + (2\theta - 1)c(n) \Rightarrow c(n) = 2^{2n-1}.$$

We postpone a detailed anlysis of the spectral distribution of $aY_taY_t^\star, \tau(a) \neq 0$ to future publications.

3.5.2 $\lambda \in (0,1], \theta = 1/2$

Recall that for general parameter values (λ,θ), $(t,z) \mapsto G_t(z)$ is a solution of the p.d.e.

$$\partial_t G_t = \partial_z\left\{[(1-2\lambda\theta)z - \theta(1-\lambda)]G_t + \lambda\theta z(z-1)G_t^2\right\}$$

with initial value $G_0(z) = 1/(z-1)$, for $t > 0$ and $z \in \mathbb{C} \setminus [0,1]$. Recall also from [12] that

$$G_\infty(z) := \frac{(2-r)z + (1/\lambda - 1) + \sqrt{r^2z^2 - Bz + C^2}}{2z(z-1)}$$

where $r = (1/\lambda\theta)$, $B = 2(r+(r-2)/\lambda), C = (1-1/\lambda)$, is the Cauchy-Stieltjes transform of the stationary distribution of the free Jacobi process (that is the spectral distribution of $PUQU^\star P$ where U is a Haar unitary variable in $\mathscr{A}$ which is $\star$-free from $\{P,Q\}$). Then

$$\partial_z\left\{[(1-2\lambda\theta)z - \theta(1-\lambda)]G_\infty + \lambda\theta z(z-1)G_\infty^2\right\} = 0$$

which can be checked in a direct way. Specializing to $\theta = 1/2, \lambda \in (0,1]$, one gets

$$\partial_t G_t = \frac{1}{2}\partial_z\left\{(1-\lambda)(2z-1)G_t + \lambda z(z-1)G_t^2\right\}$$

while

$$G_\infty(z) := \frac{(\lambda-1)(2z-1) + \sqrt{4z^2 - 4z + (1-\lambda)^2}}{2\lambda z(z-1)}$$

satisfies

$$\partial_z\left\{(1-\lambda)(2z-1)G_\infty + \lambda z(z-1)G_\infty^2\right\} = 0.$$

Set

$$H_t := G_t - G_\infty$$

then easy computations show that

$$\partial_t H_t = \frac{1}{2}\partial_z \left\{\lambda z(z-1)H_t^2 + \sqrt{4z^2 - 4z + (1-\lambda)^2} H_t\right\}.$$

Take $z \in (0,1]$ and set

$$S_t(z) := \frac{1}{z} H_t\left(\frac{1}{z}\right)$$

then

$$\partial_t S_t = -\frac{z}{2}\partial_z \left\{\lambda(1-z)S_t^2 + \sqrt{4 - 4z + (1-\lambda)^2 z^2} S_t\right\}$$

Next, we introduce

$$v_t(z) := \frac{2}{2-\lambda}\rho_{2\lambda t/(2-\lambda)}[(2-\lambda)e^{-t}\alpha(z)]$$

where we recall that

$$\rho_t(z) = \sum_{k\geq 1} \frac{1}{k} L_{k-1}^1(kt) z^k, \quad \alpha(z) = \frac{z}{(1+\sqrt{1-z})^2}.$$

Then using the non linear p.d.e

$$\partial_t \rho_t + \frac{z}{2}\partial_z \rho_t^2 = 0$$

and

$$\frac{\alpha'(z)}{\alpha(z)} = \frac{1}{z\sqrt{1-z}}$$

one derives

$$\partial_t v_t(z) + \frac{\lambda}{2} z\sqrt{1-z}\partial_z v_t^2(z) = -2e^{-t}\alpha(z)(\partial_z \rho_t)[(2-\lambda)e^{-t}\alpha(z)].$$

Accordingly, the function

$$u_t := S_t - \frac{1}{\sqrt{1-z}} v_t$$

satisfies

$$\begin{aligned}
\partial_t u_t &= -\frac{z}{2}\partial_z \left\{\lambda(1-z)u_t^2 + 2\lambda\sqrt{1-z}u_t v_t + \sqrt{4-4z+(1-\lambda)^2 z^2}u_t + \right. \\
&\quad \left. \sqrt{4+(1-\lambda)^2\frac{z^2}{1-z}}v_t\right\}(z) - \frac{1}{\sqrt{1-z}}\left[\partial_t v_t + \frac{\lambda}{2}z\sqrt{1-z}\partial_z v_t^2\right](z) \\
&= -\frac{z}{2}\partial_z \left\{\lambda(1-z)u_t^2 + 2\lambda\sqrt{1-z}u_t v_t + \sqrt{4-4z+(1-\lambda)^2 z^2}u_t + \right. \\
&\quad \left. \sqrt{4+(1-\lambda)^2\frac{z^2}{1-z}}v_t\right\}(z) + \frac{2e^{-t}\alpha(z)}{\sqrt{1-z}}(\partial_z\rho_t)[(2-\lambda)e^{-t}\alpha(z)] \\
&= -\frac{z}{2}\partial_z \left\{\lambda(1-z)u_t^2 + 2\lambda\sqrt{1-z}u_t v_t + \sqrt{4-4z+(1-\lambda)^2 z^2}u_t + \right. \\
&\quad \left. \sqrt{4+(1-\lambda)^2\frac{z^2}{1-z}}v_t\right\}(z) + 2ze^{-t}\alpha'(z)(\partial_z\rho_t)[(2-\lambda)e^{-t}\alpha(z)]
\end{aligned}$$

Denote

$$r_t(z) := \sqrt{4+(1-\lambda)^2\frac{z^2}{1-z}}$$

then

$$-\frac{z}{2}\partial_z(r_t v_t)(z) = -\frac{(1-\lambda)^2}{4r_t(z)}\frac{z^2(2-z)}{(1-z)^2}v_t(z) - zr_t(z)\alpha'(z)e^{-t}(\partial_z\rho_t)[(2-\lambda)e^{-t}\alpha(z)].$$

Rearranging terms, one gets

$$\begin{aligned}
\partial_t u_t &= -\frac{z}{2}\partial_z\left\{\lambda(1-z)u_t^2 + 2\sqrt{1-z}u_t v_t + \sqrt{4-4z+(1-\lambda)^2z^2}u_t\right\} \\
&\quad + \frac{(1-\lambda)^2}{4r_t(z)}\frac{z^2(z-2)}{(1-z)^2}v_t(z) + ze^{-t}(r_t(z)-2)\alpha'(z)(\partial_z\rho_t)[(2-\lambda)e^{-t}\alpha(z)].
\end{aligned}$$

Since

$$\lim_{\lambda\to 1} r_t(z) = 2$$

then one can see that

$$Z_t := \frac{(1-\lambda)^2}{4r_t(z)}\frac{z^2(z-2)}{(1-z)^2}v_t(z) + ze^{-t}(r_t(z)-2)\alpha'(z)(\partial_z\rho_t)[(2-\lambda)e^{-t}\alpha(z)] \to 0$$

as $\lambda \to 1$. As a matter of fact, one can expand

$$Z_t = (1-\lambda)\sum_{n=1}^{\infty} d_n(t)z^n$$

for some coefficients $d_n(t)$ depending on t, λ. Now assume

$$u_t(z) = \sum_{n=1}^{\infty} c_n(t) z^n$$

and expand

$$\sqrt{4 - 4z + (1-\lambda)^2 z^2} = \sum_{n=0}^{\infty} \gamma_n z^n$$

where

$$\gamma_n = 2 \sum_{k=0}^{n} \frac{\beta_k \beta_{n-k}}{z_1^k z_2^k} z^k$$

with

$$\beta_k = \frac{-(2n)!}{(2n-1)(2^n n!)^2}.$$

Then it is easy to see that

$$c_1'(t) = -c_1(t) + (1-\lambda) d_1(t).$$

But

$$u_0(z) = M_0(z) - M_\infty(z) - \frac{1}{\sqrt{1-z}} v_0(z)$$

and

$$\begin{aligned}
\frac{1}{\sqrt{1-z}} v_0(z) &= \frac{2}{2-\lambda} \sum_{n=1}^{\infty} \frac{z^n}{2^{2n}} \sum_{k=1}^{n} \binom{2n}{n-k} (2-\lambda)^k \\
M_\infty(z) &= \frac{(1-\lambda)(z-2) + \sqrt{4 - 4z + (1-\lambda)^2 z^2}}{2\lambda(1-z)}.
\end{aligned}$$

Hence

$$c_1(0) = \frac{\lambda - 1}{2\lambda} + \frac{1}{2\lambda} - \frac{1}{2} = 0$$

which entails that

$$c_1(t) = (1-\lambda) \left(\int_0^t d_1(s) e^s ds \right) e^{-t}.$$

It is also easy to see that $d_1(t) = 0$ so that $c_1(t) = 0$.

Chapitre 4

Spectral distribution of large-size Brownian motions on the complex linear group

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Abstract :
We show how the approach used in [13] applies to describe the large-size limit of the marginal distribution of the Brownian motion on the complex linear group.

4.1 Overview

For $t \in \mathbb{R}$, let

$$m_n(t) := \frac{e^{-nt/2}}{n} \sum_{k=0}^{n-1} \frac{(-tn)^k}{k!} \binom{n}{k+1},\ n \geq 1,\ m_0(t) = 1. \tag{4.1.1}$$

Though at a first glance $m_n(t)$ does not reveal any special feature, it is rather connected to interesting and different objects according to whether $t \geq 0$ or $t \leq 0$. Indeed, if $(Y_t)_{t\geq 0}$ is a standard right-invariant Brownian motion on $U(d)$ [1] ([29]), then for any $t \geq 0$

$$m_n(t) = \lim_{d\to\infty} \frac{1}{d}\mathbb{E}(\mathrm{tr}(Y_{t/d}^n)), n \in \mathbb{Z}.$$

1. This is not a restriction since $Y^\star = Y^{-1}$ is a left-invariant Brownian motion on $U(d)$.

where $m_{-n}(t) := m_n(t), n \geq 0$. This result, proved independently in [5], [36] and [42], was strenghtened in [28] where a full expansion of the averaged power sums of Y_t is obtained. There, the integers

$$n^{k-1}\binom{n}{k+1},\ k = 0, 1, \ldots, n-1,$$

were given a nice combinatorial interpretation by means of certain paths in the Cayley graph of the symmetric group. As to the representative probability distribution, say μ_t, it was described in [7] and reveals a phase transition at $t = 4$. More precisely, its support covers the whole torus $\mathbb{T}$ exactly at that time and remains so afterwards. Note that this phase transition is also present in the decay of $m_n(t)$ in n which switches from polynomial to exponential orders ([28]). When $t \leq 0$, it was proved in [5] that

$$m_n(t) = \lim_{d\to\infty} \frac{1}{d}\mathbb{E}(\mathrm{tr}[(Z_{-t/d}^{\star}Z_{-t/d})^n]),\ n \geq 0 \tag{4.1.2}$$

where $(Z_s)_{s\geq 0}$ is a right-invariant Brownian motion [2] on the linear complex group $GL(d, \mathbb{C})$ ([29], [34]). The corresponding probability distribution, say ν_t, was described in [7]. In particular, ν_t is compactly-supported in the positive half-line and it support spreads as t decreases. Moreover, Corollary 10 in [38] provides a realization of a non commutative random variable with spectral distribution ν_t while [21] shows that ν_t is the central limiting distribution for the product of free identically-distributed positive random variables ([21]).

Recently, the description of μ_t was revisited ([13]). There, we started from a residue-type integral representation of $m_n(t)$ then proved that for any $t \in (0, 4)$, there exists a unique Jordan curve ρ_t satisfying

- $\rho_t \cap [1, \infty[= \emptyset$,
- If
$$h_t(z) := (1-z)e^{t(1/z-1/2)},$$
then $h_t(\rho_t) \in \mathbb{T}$.

The first condition imposed on γ_t allows to use an analytic branch of the function $z \mapsto \log(1-z)$, which in turn allows to perform an integration by parts in the integral representation. The second condition together with some computations lead to the description of $\mu_t, t < 4$. Besides, ρ_t is precisely built upon two Jordan curves that intersect up to $t = 4$ and disconnect afterwards, providing another feature of the phase transition evoked above. In this note, we show how this approach may be adapted in order to retrieve the description of ν_t. Compared to [13], the integral representation of $m_n(t)$ we need when $t < 0$ involves $z \mapsto h_t(-1/z)$ rather than $z \mapsto h_t(z)$ when $t > 0$. In this way, the essential singularity at the origin present when $t > 0$ becomes a multiple pole of order n when $t < 0$.

2. This is not a restriction since $(Z^{-1})^{\star}$ is a left-invariant Brownian motion on $GL(d, \mathbb{C})$.

4.2 Description of $\nu_t, t < 0$ revisited

4.2.1 An integral representation

Since

$$\int_\gamma e^{-ntz}\frac{dz}{z^k} = \frac{(-nt)^{k-1}}{(k-1)!}, \, 1 \leq k \leq n,$$

and vansihes for $k = 0$ then

$$m_n(t) = \frac{e^{-nt/2}}{2i\pi n}\int_\gamma e^{-ntz}\left(1+\frac{1}{z}\right)^n dz.$$

This is nothing else but one of the numerous integral representations of Laguerre polynomials ([39]) and already appeared in [42] when $t > 0$. Now, we already know that ν_t is compactly supported in the positive half-line. Hence, it suffices to prove that for each $t < 0$ there exists a unique Jordan curve around the origin γ_t such that

$$g_t(z) := e^{-t(z+(1/2))}\left(1+\frac{1}{z}\right) \in \mathbb{R}$$

for any $z \in \gamma_t$. The proof of this claim is a technical exercise from real analysis that we solve below.

4.2.2 Existence of γ_t

Write $z = x + iy$, then

$$g_t(z) \in \mathbb{R} \quad \Leftrightarrow \quad y\cos(ty) + (x^2 + y^2 + x)\sin(ty) = 0. \tag{4.2.3}$$

Since any real number z satisfies (4.2.3), then we rather focus on

$$-y^2 - y\cot(ty) = x^2 + x \tag{4.2.4}$$

which reduces the set of real solutions of (4.2.4) to

$$x_t^{\pm}(0) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{t}}.$$

Since $\cot(\pm\pi) = \infty$, then (4.2.4) is satisfied as soon as one finds a positive number $0 < y_t < -\pi/t$ such that

$$\frac{1}{4} - y^2 - y\cot(ty) \geq 0$$

for all $|y| \leq y_t$ (y lies in a symmetric interval around the origin since (4.2.4) is invariant by $z \mapsto \overline{z}$). The curve γ_t we are looking for would be defined by

$$x_t^{\pm}(y) = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - y^2 - y\cot(ty)},\ |y| \leq y_t.$$

Note that, up to elementary transformations, γ_t fits the boundary of the region Ω_t described p.271 in [7]. Indeed, the changes $y \to y/2, x \to (x-1)/2$ and the 'time change' $t \to -4t$ leads to

$$2y\cot(2ty) = x^2 + y^2 - 1.$$

4.2.3 Existence of y_t

Set

$$f_t(y) := 2y\sin(ty) + \cos(ty),\ y \in (\pi/t, -\pi/t),$$

then for any $y \neq 0$

$$\frac{1}{4} - y^2 - y\cot(ty) \geq 0 \quad \Leftrightarrow \quad |f_t(y)| \leq 1.$$

Let $y \in [0, -\pi/t)$, then the inequality $\sin u \geq u, u \leq 0$ shows that

$$y \mapsto (ty)\cot(ty)$$

is decreasing. But

$$itf_t'(y) = 2\sin(ty)\left[ty\cot(ty) - \frac{t-2}{2}\right]$$

thus there exists $a_t \in [-\pi/(2t), -\pi/t[$ such that $f_t'(a_t) = 0$ and

y	0		a_t		$-\pi/t$
$f_t'(y)$	$-$		0		$+$
$f_t(y)$	1	$\searrow$	$f_t(a_t)$	$\nearrow$	-1

Note that the very definition of a_t implies that

$$f_t(a_t) = \frac{\sin(ta_t)}{ta_t}[2ta_t^2 + \frac{t}{2} - 1] \leq 2ta_t^2 + \frac{t}{2} - 1 < -1$$

where we used the inequality $\sin(u) \geq u, u < 0$. Consequently, there exists $0 < y_t < a_t$ such that $|f_t(y)| \leq 1$ for all $y \in [-y_t, y_t]$.

Remark 4.2.1. *It is easy to see that $g_t(\gamma_t)$ lies in the positive half-line. Indeed, write $z = x + iy \in \gamma_t$ then*

$$\begin{aligned} g_t(z) = \Re(g_t(z)) &= \tfrac{e^{-t(x+(1/2))}}{x^2+y^2} \left[(x^2 + y^2 + x)\cos(ty) - y\sin(ty)\right] \\ &= y\tfrac{e^{-t(x+(1/2))}}{x^2+y^2} \left[\cot(-ty)\cos(ty) + \sin(-ty)\right] \\ &= \tfrac{y}{\sin(-ty)} \tfrac{e^{-t(x+(1/2))}}{x^2+y^2} \\ &= \tfrac{y}{\sin(ty)} \tfrac{e^{-t(x+(1/2))}}{y\cot(ty)+x}. \end{aligned}$$

which is obviously positive.

4.2.4 Monotonicity of $y \mapsto g_t(x_t^{\pm}(y), y)$

Set

$$k_t(x, y) := \frac{y}{\sin(ty)} \frac{e^{-tx}}{y\cot(ty) + x} = e^{t/2} g_t(x, y)$$

and

$$v_t(y) := \sqrt{\frac{1}{4} - y^2 - y\cot(ty)}, \quad y \in [0, y_t],$$

then

$$\partial_y g_t[x_t^{\pm}(y), y] = \partial_y x_t^{\pm}(y)\partial_x(k_t)[x_t^{\pm}(y), y] + \partial_y(k_t)[x_t^{\pm}(y), y]$$

where

$$\begin{aligned} \partial_x(k_t)(x, y) &= -\frac{ye^{-tx}}{(x + y\cot(ty))^2 \sin(ty)}[t(x + y\cot(ty)) + 1] \\ \partial_y(k_t)(x, y) &= \frac{e^{-tx}}{(x + y\cot(ty))^2 \sin(ty)}[(ty^2 + x) - txy\cot(ty)] \\ \partial_y(x_t^{\pm})(y) &= \mp\frac{4y\sin^2(ty) + \sin(2ty) - 2ty}{4\sin^2(ty)v_t(y)}. \end{aligned}$$

Observe that the analysis performed in the previous paragraph shows that $v_t(y) = 0, y \in [0, y_t]$ if and only if $y = y_t$. Besides $\partial_y(x_t^{\pm})(0) = 0$ and the inequality $\sin u \geq u, u \leq 0$ shows again that x_t^+ (resp. x_t^-) is decreasing (resp. increasing) from $(0, y_t)$ onto $(-1/2, x_t^+(0))$ (resp. onto $(x_t^-(0), -1/2)$). Note also that (4.2.4) shows that

$$x + y\cot(ty) = -(y^2 + x^2)$$

along the curve γ_t. It follows that

$$\partial_x(k_t)(x_t^{\pm}(y), y) \geq 0.$$

Now, $\partial_y(k_t)(x_t^-(y),y)$ is positive on $(0,y_t)$ since

$$x_t^-(y) < 0 \quad \text{and} \quad 1 - ty\cot(ty) > 0.$$

Consequently, the map $y \mapsto g_t(x_t^-(y),y)$ is increasing. Also, we can prove after tedious computations that $y \mapsto g_t(x_t^+(y),y)$ is decreasing. In fact, writing

$$\begin{aligned}\partial_y g_t[x_t^+(y),y] = &\ \tfrac{e^{-t/2-tx_t^+(y)}}{y\cot(ty)+x_t^+(y)}\partial_y\left[\tfrac{y}{\sin(ty)}\right] + \tfrac{ye^{-t/2}\partial_y[e^{-tx_t^+(y)}]}{(y\cot(ty)+x_t^+(y))\sin(ty)}\\ &+\tfrac{ye^{-t/2-tx_t^+(y)}}{\sin(ty)}\partial_y\left[\tfrac{1}{y\cot(ty)+x_t^+(y)}\right]\end{aligned}$$

we get

$$\begin{aligned}\partial_y g_t[x_t^+(y),y] = &\ \Big[\tfrac{(\sin(ty)-ty\cot(ty))}{(y\cot(ty)+x_t^+(y))\sin^2(ty)} + \tfrac{-ty\partial_y[x_t^+(y)]}{(y\cot(ty)+x_t^+(y))\sin(ty)}\\ &+\tfrac{y(-\cot(ty)+ty/\sin^2(ty)-\partial_y[x_t^+(y)])}{(y\cot(ty)+x_t^+(y))^2\sin(ty)}\Big]e^{-t/2-tx_t^+(y)}\end{aligned}$$

which can be written

$$\begin{aligned}\partial_y g_t[x_t^+(y),y] = &\ \Big[\tfrac{(\sin(ty)-ty\cot(ty)-ty\partial_y[x_t^+(y)]\sin(ty))}{(y\cot(ty)+x_t^+(y))\sin^2(ty)}\\ &+\tfrac{-y\cot(ty)+ty^2/\sin^2(ty)-y\partial_y[x_t^+(y)]}{(y\cot(ty)+x_t^+(y))^2\sin(ty)}\Big]e^{-t/2-tx_t^+(y)}.\end{aligned} \tag{4.2.5}$$

But since

$$\partial_y[x_t^+(y)] = \frac{-4y\sin^2(ty)-\sin(2ty)+2ty}{4(1/2+x_t^+(y))\sin^2(ty)}$$

then

$$\begin{aligned}&-y\cot(ty)+ty^2/\sin^2(ty)-y\partial_y[x_t^+(y)]\\ &=\frac{2x_t^+(y)(-y\sin(2ty)+2ty^2)+4y^2\sin^2(ty)}{4(1/2+x_t^+(y))\sin^2(ty)}.\end{aligned}$$

Next, writing $y^2 = -x_t^+(y)^2 - x_t^+(y) - y\cot(ty)$, we get

$$\frac{2x_t^+(y)(-y\sin(2ty)+2ty^2)-4\sin^2(ty)(x_t^+(y)^2+x_t^+(y))-2y\sin(2ty)}{4(1/2+x_t^+(y))\sin^2(ty)}.$$

Gathering terms proportional to $\sin(2ty)$, we are led to

$$\frac{-2y(x_t^+(y)+1)\sin(2ty)+4ty^2x_t^+(y)-4\sin^2(ty)(x_t^+(y)^2+x_t^+(y))}{4(1/2+x_t^+(y))\sin^2(ty)}$$

and finally to

$$\frac{[-(x_t^+(y)+1)\sin^2(ty)+ty^2][y\cot(ty)+x_t^+(y)]-ty^3\cot(ty)}{(1/2+x_t^+(y))\sin^2(ty)}.$$

Accordingly,

$$\partial_y g_t = \Big[\frac{-ty\sin(2ty)(2x_t^+(y)+1)+4ty^2\sin^2(ty)+ty\sin(2ty)-2t^2y^2-2\sin^2(ty)+4ty^2}{2(2x_t^+(y)+1)(y\cot(ty)+x_t^+(y))\sin^3(ty)}$$
$$-\frac{4ty^3\cot(ty)}{2(2x_t^+(y)+1)(y\cot(ty)+x_t^+(y))^2\sin^3(ty)}\Big] e^{-t/2-tx_t^+(y)}.$$

Using the fact that

$$\frac{4ty^3\cot(ty)}{y\cot(ty)+x_t^+(y)} = 4ty^2\cos^2(ty) - 2ty(x_t^+(y)+1)\sin(2ty),$$

we obtain

$$\partial_y g_t = \Big[\frac{-ty\sin(2ty)(2x_t^+(y)+1)+4ty^2\sin^2(ty)+ty\sin(2ty)-2t^2y^2-2\sin^2(ty)+4ty^2}{2(2x_t^+(y)+1))\sin^3(ty)(y\cot(ty)+x_t^+(y))}$$
$$-\frac{4ty^2\cos^2(ty)-2ty(x_t^+(y)+1)\sin(2ty)}{2(2x_t^+(y)+1)\sin^3(ty)(y\cot(ty)+x_t^+(y))}\Big] e^{-t/2-tx_t^+(y)}.$$

Which can be reduced to

$$\left[\frac{-1+4ty^2-2t^2y^2-(4ty^2-1)\cos(2ty)+2ty\sin(2ty)}{2(2x_t^+(y)+1)\sin^3(ty)(y\cot(ty)+x_t^+(y))}\right] e^{-t/2-tx_t^+(y)}.$$

Consequently, $\partial_y g_t[x_t^+(y), y] \leq 0$ if and only if

$$-1+4ty^2-2t^2y^2-(4ty^2-1)\cos(2ty)+2ty\sin(2ty) \geq 0. \tag{4.2.6}$$

Performing the variables change

$$(s,x) = (\frac{-1}{t}, -ty)$$

we recover the reduced form

$$1+2x^2+4sx^2-(1+4sx^2)\cos(2x)-2x\sin(2x) \geq 0.$$

Now setting, for $s \geq 0$ and $x \in (0,\pi)$, the function

$$h(s,x) = 1+2x^2+4sx^2-(1+4sx^2)\cos(2x)-2x\sin(2x)$$

and differentiating h with respect to s, we get

$$\partial_s h(s,x) = 4x^2 - 4x^2\cos(2x) \geq 0,$$

for any $x \in (0,\pi)$. Which yields

$$h(s,x) \geq h(0,x), \text{ for any } s > 0 \text{ and any } x \in (0,\pi).$$

Next, differentiating the function $x \mapsto h(0,x)$ for $x \in (0,\pi)$, yields

$$\partial_x h(0,x) = 4x^2 - 4x\cos(2x) \geq 0.$$

Consequently,

$$h(0,x) \geq 0 = h(0,0)$$

for any $x \in (0,\pi)$. This gives the inequality (4.2.6) and thus $y \mapsto g_t[x_t^+(y), y]$ is decreasing.

4.2.5 Description of ν_t

By Cauchy's residue Theorem :

$$m_n(t) \quad = \tfrac{1}{2i\pi n} \int_{\gamma_t} [g_t(z)]^n dz$$

which transforms after integrating by parts to

$$m_n(t) \quad = -\tfrac{1}{2i\pi} \int_{\gamma_t} [g_t(z)]^n z \tfrac{\partial_z g_t(z)}{g_t(z)} dz.$$

Split $\gamma_t = \gamma_t^+ \cup \gamma_t^-$ where

$$\gamma_t^{\pm} := \{z_t^{\pm}(y) := x_t^{\pm}(y) + iy,\ |y| \leq y_t\}$$

so that

$$\begin{aligned}\int_{\gamma_t} [g_t(z)]^n z \frac{\partial_z g_t(z)}{g_t(z)} dz \quad &= \textstyle\int_{-y_t}^{y_t} [g_t(z_t^+(y))]^n z_t^+(y) \frac{\partial_y [g_t(z_t^+)](y)}{g_t(z_t^+(y))} dy \\ &- \textstyle\int_{-y_t}^{y_t} [g_t(z_t^-(y))]^n z_t^-(y) \frac{\partial_y [g_t(z_t^-)](y)}{g_t(z_t^-(y))} dy.\end{aligned}$$

But since $z_t^+(-y) = \overline{z_t^+(y)}, g_t(z_t^+(y)) \in \mathbb{R}$ then

$$g_t(z_t^+(-y)) = g_t(z_t^+(y))$$

and

$$\begin{aligned}&\int_{-y_t}^{y_t} [g_t(z_t^+(y))]^n z_t^+(y) \frac{\partial_y [g_t(z_t^+)](y)}{g_t(z_t^+(y))} dy \\ &= \int_{-y_t}^{y_t} [g_t(z_t^+(y))]^n z_t^+(y) \partial_y [\log g_t(z_t^+)](y) dy \\ &= \int_0^{y_t} [g_t(z_t^+(y))]^n z_t^+(y) \partial_y [\log g_t(z_t^+)](y) dy - \int_0^{y_t} [g_t(z_t^+(y))]^n \overline{z_t^+(y)} \partial_y [\log g_t(z_t^+)](y) dy \\ &= 2i \int_0^{y_t} [g_t(z_t^+(y))]^n \mathrm{Im}\,[z_t^+(y)] \partial_y [\log g_t(z_t^+)](y) dy.\end{aligned}$$

Similarly

$$\begin{aligned}&\int_{-y_t}^{y_t} [g_t(z_t^-(y))]^n z_t^-(y) \frac{\partial_y [g_t(z_t^-)](y)}{g_t(z_t^-(y))} dy \\ &= 2i \int_0^{y_t} [g_t(z_t^-(y))]^n \mathrm{Im}\,[z_t^-(y)] \partial_y [\log g_t(z_t^-)](y) dy.\end{aligned}$$

As a result

$$\frac{1}{2i\pi}\int\limits_{\gamma_t^{\pm}}[g_t(z)]^n z\frac{\partial_z g_t(z)}{g_t(z)}dz \quad = \tfrac{1}{\pi}\left\{\int_0^{y_t}[g_t(z_t^+(y))]^n\mathrm{Im}\,[z_t^+(y)]\partial_y[\log g_t(z_t^+)](y)dy\right.$$

$$\left. - \int_0^{y_t}[g_t(z_t^-(y))]^n\mathrm{Im}\,[z_t^-(y)]\partial_y[\log g_t(z_t^-)](y)dy\right\}.$$

Finally, the monotonicity of $y \mapsto g_t(z_t^{\pm}(y))$ entails

$$m_n(t) = \frac{1}{\pi}\int\limits_{g_t(x_t^-(0))}^{g_t(x_t^+(0))} x^n\mathrm{Im}\,[g_t^{-1}(x)]\frac{dx}{x}$$

where

$$g_t(x_t^{\pm}(0)) = \left[1-\frac{t}{2}\pm\sqrt{\frac{t^2}{4}-t}\right]\exp\{\pm\sqrt{\frac{t^2}{4}-t}\}.$$

Using the time change $t \to -4t$, the interval $[g_t(x_t^-(0)), g_t(x_t^+(0))]$ transforms into the support of ν_t written in [5], Proposition 11.

Bibliographie

[1] S. Attal, A. Dhahri. *Repeated Quantum Interactions and Unitary Random Walks.* J. Theoret. Prob. 23(2), pp 345-361 2010.

[2] S. Attal, M. Émery. *Equation de structure pour des martingales vectorielles.* Séminaire de Probabilités, XXIX, volume 1583 of Lecture Notes in Math., pages 256-278. Springer, Berlin, 1994.

[3] D. Bakry. *Remarques sur les semi-groupes de Jacobi. Hommage à P. André Meyer et J. Neveu. Astérisque.* **236**, 1996, 23-39.

[4] F. Benaych-Goerges, T. Lévy. *A continuous semigroup of notions of independence between the classical and the free one. Ann. Probab.* **39**, no. 3, 2011, 904-938.

[5] P. Biane. *Free Brownian motion, free stochastic calculus and random matrices. Fields. Inst. Commun.*, **12**, Amer. Math. Soc. Providence, RI, 1997. 1-19.

[6] P. Biane. *Processes with free increments. Math. Z*, **227**, 1998, no. 1. 143-174.

[7] P. Biane. *Segal-Bargmann transform, functional calculus on matrix spaces and the theory of semi-circular and circular systems. J. Funct. Anal.* **144**. 1997, no. 1, 232-286.

[8] M. Capitaine, M. Casalis. *Asymptotic freeness by generalized moments for Gaussian and Wishart Matrices. Application to Beta random matrices. Indiana Univ. Math. J.* **53**, no. 2 , 2004, 397-431.

[9] B. Collins. *Product of random projections, Jacobi ensembles and universality problems arising from free probability. Probab. Theor. Rel. Fields.* **133**, no. 3, 2005, 315-344.

[10] R. C. Dalang, A. Morton, and W. Willinger, *Equivalent martingale measures and no-arbitrage in stochastic securities market models.* Stochastics Stochastics Rep. 29 (1990), no. 2, 185–201.

[11] N. Demni, M. Zani. *Large deviations for statistics of Jacobi process. Stoch. Proc. Appl.* **119**, 2009, 518-533.

[12] N. Demni. *Free Jacobi process. J. Theo. Probab.* **21**, no.1. (2008), 118-143.

[13] N. Demni, T. Hmidi. *Spectral Distribution of the Free unitary Brownian motion : another approach. Sém. Probab. XLIV.* 2012. 191-206.

[14] Y. Doumerc. *Matrix Jacobi Process. Ph. D. Thesis*. Paul Sabatier Univ. May 2005.

[15] K. J. Dykema, A. Nica, D. V. Voiculescu. *Free Random Variables. CRM Monograph Series, 1*. 1992.

[16] M. Émery. *On the Azéma martingales*. Séminaire de Probabilités, XXIII, volume 1372 , pages 66-87. Springer 1989.

[17] M. Émery. *A discrete approach to the chaotic representation property*. Séminaire de Probabilités, XXXV, volume 1755 of Lecture Notes in Mathematics, pages 123-138. Springer, Berlin, 2001.

[18] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi. *Higher Transcendental Functions*. **Vol I**. *McGraw-Hill, New York*. 1981.

[19] H. Föllmer and A. Schied. *Stochastic finance*. Volume 27 of de Gruyter Studies in mathematics. Walter & Co., Berlin, 2004.

[20] F. Hiai, D. Petz. *The Semicircle Law, Free Random Variables and Entropy*. Mathematical Surveys and Monographs. Vol 77. A. M. S.

[21] K. P. Ho. *Central limit for the products of free random variables. Available on arXiv*.

[22] H. Holden, T. Lindstrøm, B. Øksendal, and J. Ubøe. *Discrete Wick calulus and stochastic functional equations*. Potential Analysis, 1 :291-306, 1992.

[23] C. Houdré and N. Privault. *Concentration and deviation inequalities in infinite dimensions via covariance representations*. Bernoulli, 8(6) :697-720, 2002.

[24] K. Itô, *Multiple Wiener integral*, J. Math. Soc. Japan 3 (1951), 157-169. MR 0044064 (13,364a)

[25] J.Jacod, A.N. Shiryaev. *Local martingales and fundamental asset pricing theorems in the discrete-time case*. Finance Stoch. 2, No.3, 259-273 (1998).

[26] I. Karatzas and D. Ocone. *A generalized Clark representation formula, with application to optimal portfolios*. Stochastics Stochastic Rep.34, 187-220 (1991).

[27] M. Leitz-Martini, *A discrete Clark–Ocone formula*, Maphysto Research Report No 29, 2000.

[28] T. Lévy. *Schur-Weyl duality and the heat kernel measure on the unitary group. Adv. Math.* **218**, 2008, no. 2, 537-575.

[29] M. Liao. *Lévy Processes in Lie Groups. Cambridge Tracts in Mathematics*. 2004.

[30] P.A. Meyer. *Éléments de probabilités quantiques*. Séminaire de Probabilités XX, Lecture Notes in Mathematics 1204,. Springer 1986.

[31] P.A. Meyer. *Diffusions quantiques (exposé en trois parties)*. Séminaire de Probabilités, XXIV, pages 370-396. Springer 1990.

[32] P.A. Meyer. *Quantum Probability for Probabilists*. Lecture Notes in Mathematics 1538,. Springer, Berlin, 1993.

[33] A. Nica, R. Speicher. *Lectures on Combinatorics of Free Probability. London Mathematical Society Lecture Note Series, 335.*

[34] J. R. Norris, L. C. G. Rogers, D. Williams. *Brownian motions of ellipsoids. Trans. Amer. Math. Soc.* **294**, 1986, no. 2, 757-765.

[35] N. Privault. *Stochastic analysis of Bernoulli processs*. Proba. Surv. 5 (2008).

[36] E. M. Rains. *Combinatorial Properties of Brownian Motion on the Compact Classical Groups. J. Theor. Probab.* **10**, no. 3. 1997, 659-679.

[37] E. D. Rainville. *Special functions. The Macmillan Co. New York*. 1960.

[38] N. Sakuma, H. Yoshida. *New limit theorems related to free multiplicative convolution*. it Stud. Math. **214**, 2013, No. 3, 251-264.

[39] G. Szegö. *Orthogonal Polynomials. Fourth Edition, Amer. Math. Soc., Providence,* 1975.

[40] D. V. Voiculescu. *The analogues of entropy and of Fisher's information measure in free probability theory*. VI. Liberation and mutual free information. *Adv. Math.* **146**, no. 2. 1999, 101-166.

[41] N. Wiener, *The Homogeneous Chaos*, Amer. J. Math. 60 (1938), 897-936. MR 1507356

[42] F. Xu. *A random matrix model from two-dimensional Yang-Mills theory. Comm. Math. Phys.* **190**, 1997, no. 2, 287-307.

Résumé

Mon travail de thèse est composé de deux parties bien distinctes, la première partie est consacrée à l'analyse stochastique en temps discret des marches aléatoires obtuses quant à la deuxième partie, elle est liée aux probabilités libres. Dans la première partie, on donne une construction des intégrales stochastiques itérées par rapport à une famille de martingales normales d-dimentionelles. Celle-ci permet d'étudier la propriété de représentation chaotique en temps discret et mène à une construction des opérateurs gradient et divergence sur les chaos de Wiener correspondant. On obtient, en particulier, une interprétation probabiliste de l'opérateur gradient comme opérateur de différence finie et on montre que l'opérateur divergence coïncide avec l'intégrale stochastique sur les processus prévisibles de carré sommables. L'opérateur gradient est utilisé ensuite pour démontrer une formule de représentation prévisible de type Clark-Ocone. Cette formule permet, en particulier, d'obtenir deux identités de covariance qu'on utilise avec l'interprétation de l'opérateur gradient afin de prouver une inégalité de déviation pour les fonctionnelles des marches aléatoires obtuses. La fin de cette partie est consacrée à l'étude du problème de couverture des options en marché complet comme application de la formule de Clark-Ocone. Dans la deuxième partie, on s'intéresse dans un premier temps à l'étude de la mesure spectrale du processus de Jacobi libre $(J_t)_{t\geq 0}$. Ce dernier est défini pour tout instant t par $J_t \triangleq PY_tQY_t^{\star}P$ où $\{P,Q\}$ sont deux projecteurs et $(Y_t)_{t\geq 0}$ est un MB unitaire libre qui sont $\star$-libre dans une algèbre de von Neumann $\mathcal{A}$. On calcule, en paticulier, la loi du processus $(J_t)_{t\geq 0}$ partant de l'identité de l'algèbre de von Neumann compressée $P\mathcal{A}P$ où il prend ses valeurs quand le rang des projecteurs vaut 1/2. Celle-ci coïncide avec la loi du cosinus du MB unitaire libre modulo le changement de temps $t \to 2t$. Deux preuves de ce résultat sont présentées : la première repose sur la résolution explicite d'une EDP non linéaire alors que la deuxième est de nature combinatoire. Dans un second temps, on a revisité la description de la mesure spectrale de la partie radiale du mouvement Brownien sur $Gl(d,\mathbb{C})$ quand $d \to +\infty$. Biane a démontré que cette mesure est absolument continue par rapport à la mesure de Lebesgue et que son support est compact dans $\mathbb{R}_+$. Notre contribution consiste à redémontrer le résultat de Biane en partant d'une représentation intégrale de la suite des moments sur une courbe de Jordon autour de l'origine et moyennant des outils simples de l'analyse réelle et complexe.

Mots clés : **Marche aléatoire obtuse, Martingale normale, Intégrale stochastique, Temps discret, Calcul chaotique, Mouvement Bownien unitaire libre, Processus de Jacobi libre, Mesure spectrale, Théorème des résidues de Cauchy.**

Abstract

My PhD work is composed of two parts, the first part is dedicated to the discrete-time stochastic analysis for obtuse random walks as to the second part, it is linked to free probability. In the first part, we present a construction of the stochastic integral of predictable square-integrable processes and the associated multiple stochastic integrals of symmetric functions on $\mathbb{N}^n$ $(n \geq 1)$, with respect to a normal martingale. This allows to study the chaotic representation property for a family of d-dimensional discrete-time martingales that do not have independent increments. In this framework, we determine the action of the gradient operator and the associated Clark-Ocone representation formula. Applications to Poincaré inequalities, covariance identities, deviation inequalities, financial hedging, are then derived. In the second part, we are interested in a first time to the study of the spectral measure of the free Jacobi process (hereafter J). The definition of J is motivated by the asymptotic freeness of some random matrices. It can also be constructed from the free unitary Brownian motion multiplied by two suitables projections. We show that, in the special case when the projections have the same rank equal to 1/2, the spectral measure of J coincides with the cosine of the free unitary BM modulo the change of time $t \to 2t$. Two proofs of this result are presented : the first is based on the explicit resolution of a nonlinear PDE while the second is combinatorial. In a second step, we revisited the description of the marginal distribution of the Brownian motion on the large-size complex linear group. Precisely, let $(Z_t^{(d)})_{t\geq 0}$ be a Brownian motion on $GL(d,\mathbb{C})$ and consider ν_t the limit as $d \to \infty$ of the distribution of $(Z_{t/d}^{(d)})^{\star}Z_{t/d}^{(d)}$ with respect to $\mathbb{E}\times \mathrm{tr}$. Then, we give an explicit representation of

$$m_n(\nu_t) \triangleq \lim \mathbb{E}\left(\mathrm{tr}((Z_{t/d}^{(d)})^{\star}Z_{t/d}^{(d)})^n\right)$$

as an integral along a Jordan curve around the origin.

Keywords : **Obtuse random walks, Normal martingale, Stochastic integrals, Discrete time, Chaotic calculus, Free unitary Brownian motion, Free Jacobi process, Spectral measure, Cauchy's Residue Theorem.**
Mathematics Subject Classification : **60G42, 60G50, 60H15, 46L54.**